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THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

EDITED BY

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GENERALIZATION OF THE PROBLEM OF THE ROTATION OF PRISMS PRODUCING CONSTANT DEVIATION BY TWO REFRACTIONS AND ONE INTERNAL REFLECTION

By HORACE SCUDDER UHLER

While I was reading the instructive paper on "The Rotation of Prisms of Constant Deviation" recently published by W. E. Forsythe[1] a number of questions arose in my mind and revived my old interest in the subject. Among these may be mentioned: (*a*) Is the axis of rotation given by Forsythe unique, or is it only one position of the generatrix of an extended cylindrical locus? (*b*) Does the existence of the axis in question depend upon the circumstance that the angle between the faces of incidence and emergence was assumed to be 90°? (*c*) If, in general, an axis of rotation, endowed with the properties specified by Forsythe, exists for a prism having any reasonable angle between the refracting planes, what is the location of this axis with respect both to the prism and to the optic axes of the collimator and telescope? In attempting to answer these and other questions I was led to a fuller appreciation of the fact that the papers by Pellin and Broca,[2]

[1] *Astrophysical Journal*, **45**, 278, 1917.

[2] *Journal de Physique*, **8**, 314, 1899.

by myself,[1] and by Forsythe[2] deal with special cases of very general properties of constant-deviation prisms. Since the results of my analytical investigation of the generalized problem are presumably of theoretical interest to the student of geometrical optics, and as they may also be found of practical value, it seems appropriate to present them in this place.

To prepare the way for the solution of the problem of the axis of rotation it is desirable to make a few introductory remarks concerning the broad point of view which will be taken, and also to generalize the fundamental equations pertaining to the type of constant-deviation prism of which the Pellin and Broca form is a very special case. A principal section of a prism of this kind is shown, by the outline *ABCD*, in Fig. 1.

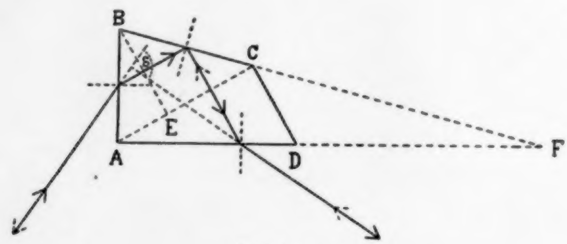

FIG. 1

In this prism $\angle DAB=90^\circ$, $\angle ABC=75^\circ$, $\angle BCD=135^\circ$, and $\angle CDA=60^\circ$. The path of a ray at minimum deviation, so called, is indicated by the broken line with the arrow-heads. The total deviation of the ray, δ, equals 90°. The particular numerical values of the angles suggest at once the usual synthesis of the rhomboidal prism from the "halves" of an ordinary equilateral prism (*ABE* and *DAC*), and the 90° total reflecting prism *BCE*. Since, in my opinion, this conception affords no help in gaining a full appreciation of the general case, but is rather a hindrance to progress, it will be entirely abandoned in the following pages. For economy of time, space, and material, the greater part of the optically ineffective extension *CFD*, of the figure *ABCD*, is omitted by the manufacturers. For the present purposes, however, it will

[1] *Physical Review*, **29**, 37, 1909. [2] *Astrophysical Journal*, **45**, 278, 1917.

be found advantageous to consider the general constant-deviation prism as having a triangular principal section. Although the path of the light is reversible, it will be convenient to designate the planes $\overline{AB}$, $\overline{BC}$, and $\overline{AD}$ (see also later diagrams) as the incidence face, the total reflection face, and the emergence face, respectively. The angles ABC and DAB will be symbolized by σ and ω, in the order named. These angles may have any values consistent with the proper operation of the prism. In my earlier paper[1] a partial generalization was introduced by allowing σ (formerly ξ) to have any feasible value, but ω was kept at 90° throughout. This amounted to studying all prisms (used as specified) giving the constant deviation 90° and equivalent to ordinary isosceles prisms having the variable refracting angle $2\tau = 2\sigma - 90°$. For well-known reasons the isosceles prisms designed for producing spectra are almost invariably given the equilateral form, so that the advance made by changing σ from 75° to a variable value was probably less important than the gain which will arise from changing ω from 90° to any reasonable value. As an immediate consequence of the variable character of ω it will be possible to design prisms giving any prescribed constant deviation while retaining all the important properties of the equivalent equilateral direct-transmission prism. The present paper, as stated above, restricts neither σ nor ω, and hence it covers all families of prisms that produce constant deviation by two refractions and one internal reflection at three different planes.

Attention will now be directed to the fundamental optical and mathematical properties of the general constant-deviation prism. In Fig. 2 let $\overline{AB}$ and $\overline{AD}$ denote respectively portions of the incidence and emergence faces. $\angle DAB = \omega$. The angles of incidence and emergence will be symbolized by ι_1 and ι_2, and they will be reckoned positive when generated by counter-clockwise rotation from their respective normals. The (total) deviation δ will be defined as the angle through which the emergent ray must be turned anti-clockwise to bring it into (vectorial) parallelism with the incident ray. Diagrams (a) and (b) show the cases where $\iota_1 > \iota_2$ and $\iota_1 < \iota_2$, respectively. In Fig. 2 (a) the emergence face, the

[1] *Physical Review*, **29**, 37, 1909.

emergent ray, and the normal at the point of emergence may be considered as constituting a rigid figure which is to be revolved counter-clockwise around the pivot A until the emergence face coincides with the incidence face. This rotation will contribute the amount ω to the value of δ. To complete the value of δ the emergent ray $\overline{OE}$ (in its new position) must be turned anti-clockwise

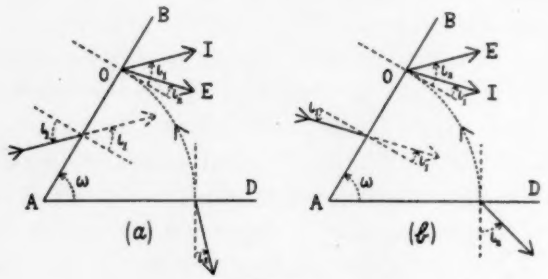

FIG. 2

through the angle EOI to the direction $\overline{OI}$ of the incident ray. The last rotation is evidently equal to $\iota_1-\iota_2$, hence $\delta=\omega+(\iota_1-\iota_2)$. In Fig. 2 (*b*) the first rotation is again equal to ω, but the second component rotation must be performed *clockwise* by an amount $\iota_2-\iota_1$. Therefore the total (anti-clockwise) deviation $\delta=\omega-(\iota_2-\iota_1)$ or $\delta=\omega+\iota_1-\iota_2$. Consequently, for any case

$$\delta=\omega+\iota_1-\iota_2. \tag{1}$$

This method of proof has been given because of its complete generality. It emphasizes the fact that when ι_1, ι_2, and ω are given, δ is uniquely determined. Since equation (1) represents a purely geometrical relation, and nothing more, it is clear that the value of the total deviation does not involve, in any manner whatsoever, the history of the ray between the occasions of incidence and emergence. Hence the range of applicability of this equation is not restricted to prismatic forms, so that the lines $\overline{AB}$ and $\overline{AD}$ of Fig. 2 need not pertain to the faces of a prism. Only when the values of ι_1 and ι_2 are not known does it become necessary to introduce quantities which depend upon the arbitrarily prescribed path of the ray between the points of incidence and emergence.

The immediate consequences of the hypotheses that, between the points of incidence and emergence, the ray shall remain in one optically homogeneous medium, and shall experience internal reflection at a plane surface, will now be considered. For the sake of generality, no reference will be made to any particular diagram. The angles of refraction associated with ι_1 and ι_2 will be denoted respectively by ρ_1 and ρ_2.

Since the segment of the ray between the points of incidence and internal reflection is straight (homogeneous, isotropic medium), a triangle will always be formed by this segment together with the sides of the angle σ. (This triangle will be finite under all circumstances except the practically impossible case where the incident ray strikes the edge common to the faces of incidence and of total reflection.) The angles of this triangle are $90°-\rho_1$, σ, and $90°-\phi_1$, where ϕ_1 denotes the angle between the ray in question and the normal to the total reflection face. Hence $180°=(90°-\rho_1)+\sigma+(90°-\phi_1)$, or

$$\rho_1=\sigma-\phi_1. \tag{2}$$

In like manner the segment of the ray between the points of internal reflection and emergence will always form a triangle when taken in conjunction with the total reflection and emergence planes. The angles of this triangle are $90°+\rho_2$, $180°-(\sigma+\omega)$, and $90°-\phi_2$, where ϕ_2 symbolizes the angle between the ray under consideration and the normal to the face of total reflection. Hence $180°=(90°+\rho_2)+(180°-\sigma-\omega)+(90°-\phi_2)$, or

$$\rho_2=\sigma+\phi_2+\omega-180°. \tag{3}$$

At internal reflection $\phi_1=\phi_2$, according to the law of reflection, therefore addition of equations (2) and (3) leads to

$$\rho_1+\rho_2=2\sigma+\omega-180°. \tag{4}$$

Like formula (1), the last equation is a purely geometrical relation involving primarily the equality of the angles ϕ_1 and ϕ_2. If now the hypothesis be made that the incidence and emergence faces of the prism are in contact with the same medium, then the condition $\iota_1=\iota_2$ (or $\rho_1=\rho_2$) leads to the equation $\rho_1=\rho_2$ (or $\iota_1=\iota_2$). This conclusion does not depend upon the form of the law of refraction:

in particular, the trigonometrical part of Snell's law is not involved. Also the index of refraction of the material of the prism relative to the surrounding medium may be less than, or equal to, unity as well as greater than this value. When $\iota_1=\iota_2$ and $\rho_1=\rho_2$, equation (1) gives $\delta=\omega$, equation (4) reduces to $\rho=\sigma+\frac{1}{2}\omega-90^\circ$, and hence equation (2) becomes $\phi=90^\circ-\frac{1}{2}\omega$. The last two equations may be thrown into more symmetrical forms by introducing the angle (σ') between the emergence and total reflection faces. Since $\sigma+\sigma'+\omega=180^\circ$ it follows that $\rho=\frac{1}{2}(\sigma-\sigma')$ and $\phi=\frac{1}{2}(\sigma+\sigma')$.

Emphasis should be laid on the fact that, when the direction of the incident beam is kept invariable while the prism is rotated around any axis perpendicular to a principal plane, each transmitted color (wave-length) will experience the constant deviation ω at the instant when $\iota_1=\iota_2$ and $\rho_1=\rho_2$. ι_1 and ι_2 are explicit functions of n [$\sin\,\iota_1=-n\cos(\sigma+\frac{1}{2}\omega)$], or implicit functions of the wave-length, whereas ρ_1 and ρ_2 have the constant value $\sigma+\frac{1}{2}\omega-90^\circ$. So far as the prism itself is concerned, the question whether the axis of the emergent beam does or does not experience pure translation normal to its direction is not germane to the properties of the prism just discussed. The additional condition that the axis of the emergent beam shall maintain a fixed position during the angular displacement of the prism will be given full consideration in later paragraphs.

As is well known, when light passes through an isosceles prism under the advantageous condition of minimum deviation, $\rho_1'=-\rho_2'=\tau$ and $\iota_1'=-\iota_2'$, where 2τ denotes the refracting angle of the prism. By identifying ρ_1' or ρ_2' with ρ an analytical definition of equivalence between the constant-deviation prism and the ordinary isosceles prism may be obtained. The equation $\rho=\frac{1}{2}(\sigma-\sigma')$ shows that ρ will be positive or negative according as σ is greater or less than σ'. Therefore ρ must be identified with ρ_1' or τ when σ exceeds σ' in value, and with ρ_2' or $-\tau$ when σ is inferior to σ'. Consequently the formula of equivalence is

$$\sigma-\sigma'=\pm 2\tau. \tag{5}$$

As $\sigma+\sigma'+\omega=180^\circ$, equation (5) may also be written

$$2\sigma+\omega=180^\circ\pm 2\tau. \tag{5'}$$

Let σ_1 and σ_2 denote the values of σ pertaining to two (supposedly) different prisms having the same value of ω and equivalent to the same direct-transmission prism characterized by τ. Then $2\sigma_1+\omega=180°+2\tau$ and $2\sigma_2+\omega=180°-2\tau$; hence, by addition, $\sigma_1+\sigma_2+\omega=180°$. Now, for one and the same constant-deviation prism (the section of which is simply a plane triangle) having the angles σ_1, σ_1', and ω the relation $\sigma_1+\sigma_1'+\omega=180°$ must hold. Comparing the last equation with the one immediately preceding, it is seen that $\sigma_1'=\sigma_2$ and so the angles σ_1 and σ_2 belong to one single triangle. Consequently the plus signs alone in formulae (5) and (5′) will lead to all constant-deviation prisms of the general type under consideration. The double sign merely means that the path of the ray through the prism has been reversed. By viewing Figs. 1 and 4 through the paper, and fixing the attention on the dotted arrow-heads, the circumstances prevailing when ρ is negative will be perceived at a glance.

The condition of equivalence just laid down is not merely a formal analytical definition. It is easy to show that the difference between the total lengths of the two rays, which lie entirely within a constant-deviation prism and which form the extreme lateral boundaries of the widest beam that can be transmitted by the prism (in the manner prescribed), is equal to $-2b\cos(\sigma+\frac{1}{2}\omega)$ or $2b\sin\tau$, where b symbolizes the actual or the effective [see Fig. 4 (c)] width of the incidence face. But $2b\sin\tau$ is precisely the expression for the width of the base of a principal section of the equivalent isosceles direct-transmission prism, the equal sides of which have each the length b. Hence, when made of the same material, the two prisms will have the same spectroscopic resolving power (Rayleigh). This result, as well as the equivalence of the magnitude of the angular dispersion, can also be obtained at once from a consideration of the nature of (internal) reflection at a plane. The chief points of difference between the optical effects produced by the two kinds of prism are: (a) greater absorption for the constant-deviation type; (b) when $\iota_1=\iota_2$ and $\rho_1=\rho_2$, the deviation effected by the isosceles prism is a function of the wave-length, whereas that produced by the internal reflection type is constant; (c) as the angle of incidence for one wave-length is varied, an algebraic

minimum of deviation may always be obtained with the isosceles form, while no stationary value can arise with the constant-deviation prism; and (*d*) the order of the spectral colors is reversed in the two cases.

With regard to the size of the angles of the constant-deviation prism the following remarks should be made. Since, when $\iota_1=\iota_2$, $\phi=90^\circ-\frac{1}{2}\omega$, it follows that ω must not be greater than $180^\circ-2\sin^{-1}\left(\frac{1}{n_r}\right)$ in order that the internal reflection may be *total* over the entire range of the spectrum within which the prism is to be used. n_r symbolizes the least value of the index of refraction pertaining to this range. (By silvering the total reflection face this limit may be raised.) When ρ_1 and ρ_2 are positive or zero, equation (4) $[\sigma+\frac{1}{2}\omega=90^\circ+\frac{1}{2}(\rho_1+\rho_2)]$ shows that $\sigma+\frac{1}{2}\omega$ cannot be acute. Similarly, $\sigma'+\frac{1}{2}\omega$ cannot be obtuse.

Before the question of the position of the axis of rotation of the constant-deviation prism is taken up, attention will be called to some interesting properties associated with equation (5′). For a given value of τ this equation may be represented by a straight line having the intercepts $90^\circ+\tau$ and $2(90^\circ+\tau)$ on the axes of σ and ω, respectively. Since there are an infinite number of points on this line in the positive quadrant, it follows that the number of constant-deviation prisms equivalent to a single isosceles prism is likewise infinite. (For obvious reasons certain pairs of values of σ and ω may not be admissible.) In particular, suppose $\tau=0$ so that, at constant deviation, $\rho_1=\rho_2=0$. Then, by Snell's law, $\iota_1=\iota_2=0$ irrespective of the value of n. Accordingly the ray enters and leaves the prism normally, and no opportunity of rotating the prism (while maintaining constant deviation) is afforded. This is the familiar condition of deviation without dispersion. When $\phi=\sigma=\sigma'=30^\circ$, $\delta=\omega=120^\circ$, but total reflection would not obtain; when $\phi=\sigma=\sigma'=45^\circ$, $\delta=\omega=90^\circ$; and when $\phi=\sigma=\sigma'=60^\circ$, $\delta=\omega=60^\circ$. The three numerical examples are illustrated in Fig. 3, outlines (*a*), (*b*), and (*c*).

When $n=1.5$ with $\tau=30^\circ$, the critical value for ϕ equals $41^\circ48'37''$. Then $\iota_1=\iota_2=48^\circ35'25''$, $\sigma=71^\circ48'37''$, and $\delta=\omega=96^\circ22'46''$. This extreme case is illustrated by Fig. 4 (*a*). When

$\tau=30^\circ$ with $\sigma=75^\circ$, $\phi=45^\circ$ and $\delta=\omega=90^\circ$. The last set of data corresponds to the original Pellin and Broca prism; see Fig. 1. When $\tau=30^\circ$ with $\sigma=90^\circ$, $\delta=\phi=\omega=60^\circ$. This is shown by outline (*b*) of Fig. 4. When $\tau=30^\circ$ with $\sigma=95^\circ$, $\phi=65^\circ$ and $\delta=\omega=50^\circ$; see (*c*) of Fig. 4. These diagrams illustrate the manner in which the optically ineffective portion of the prism changes as the angle ω decreases, τ being kept throughout at the constant value 30°. In general, for a given family of prisms (τ constant), the region containing the angle σ' is ineffective when ω is greater than $90^\circ-\tau$.

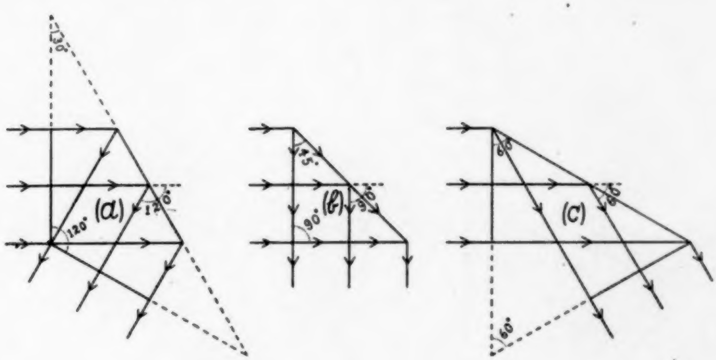

FIG. 3

When the segment of the complete ray between the points of incidence and total reflection is parallel to the emergence face, no portion of the prism is ineffective. Then $\omega=90^\circ-\tau$, $\phi=45^\circ+\frac{1}{2}\tau$, and $\sigma=45^\circ+\frac{3}{2}\tau$. Under this condition ω is always acute and ϕ is large enough to insure total reflection for all kinds of optical glass. When ω is inferior to $90^\circ-\tau$, the ineffective portion of the prism includes the angle ω. In this case the width of the transmitted beam is limited by the length of the total reflection face, whereas in the case where ω exceeds $90^\circ-\tau$ it is controlled by the breadth of the incidence face. In order that internal reflection may occur with a finite length of the total reflection face, the first internal ray must not be parallel to this face. Accordingly the following conditions must be fulfilled: $\sigma<90^\circ+\tau$ and $\omega>0^\circ$.

The transitional case,[1] $\omega = 90° - \tau$, seems to me to afford certain advantages over the original Pellin and Broca prism when $\tau = 30°$; see Fig. 4 (*b*). Some of these are: (i) No superfluous glass is involved. (ii) The prism having $\omega = 60°$ causes the telescope to be more nearly in line with the collimator than is the case when $\omega = 90°$. The former construction gives better mechanical balance to the spectroscope than the latter. Moreover, it is sometimes desirable for the observer to be farther away from a high potential discharge than is possible when the deviation is 90°. (iii) By following the

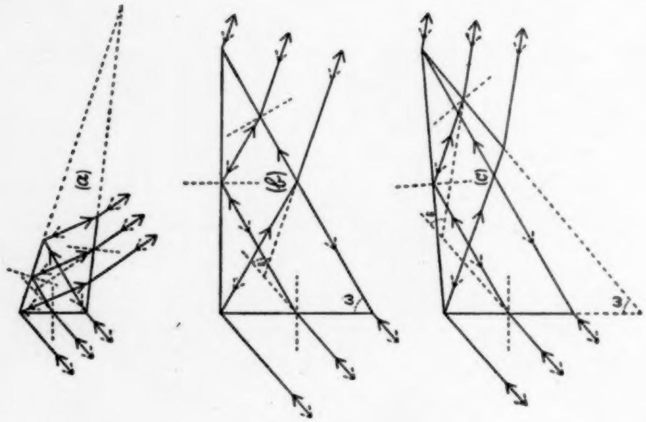

FIG. 4

paths of rays which have experienced multiple reflection, it will be seen that when $\omega = 60°$ the conditions are more favorable for minimizing the intensity of scattered light and diverting secondary spectra from the telescope than when $\omega = 90°$ and the corner (σ') is ground matt. (iv) As actually made, prisms are often not geometrically perfect near the vertical edges. When $\omega = 60°$, the prism can be made larger than is theoretically necessary for the accommodation of the incident beam of light—so as to employ only the best portions of the optical surfaces—without placing a greater demand on the stock of glass than when $\omega = 90°$; for, with

[1] I have tested this experimentally with a prism made and kindly loaned by Professor Charles S. Hastings. The results were highly satisfactory and in complete accord with the theory.

identical incidence faces, the ratio of the volume of the original Pellin and Broca prism to that of the $\omega=60^\circ$ prism is at least equal to $\frac{1}{12}(7+5\sqrt{3})\doteq 1.305$. (v) The $30^\circ-60^\circ-90^\circ$ prism can be used as a direct-transmission equilateral prism. (vi) Two $30^\circ-60^\circ-90^\circ$ prisms, of the same kind of glass, can be cemented together with their faces of intermediate area in optical contact so as to form one larger equilateral prism of relatively large resolving power. The only hypothetical disadvantage associated with $\omega=60^\circ$ to which my attention has been called involves the question of economy in manufacture. It depends on the fact that the faces of the 60° prism are somewhat larger than the homologous faces of the 90° type. More precisely, for prisms of identical incidence faces, the ratio of the area of the emergence face for $\omega=60^\circ$ to that of the least value of the like face for $\omega=90^\circ$ equals $3-\sqrt{3}\doteq 1.268$, and for the total reflection faces the ratio equals $\sqrt{2}\doteq 1.414$, at most.

Before taking up the next problem it seems desirable to state that no attempt has been made in the preceding paragraphs to follow out in detail the paths of rays which pass through the constant-deviation prism when ι_1 and ρ_1 are respectively not equal to ι_2 and ρ_2. In other words, the entire field of view in the telescope on both sides of the optic axis (line of "minimum" or constant deviation) has not been considered. The investigation of this matter would involve Snell's law and a knowledge of the values of the indices of refraction of the material of the prism for the extreme wave-lengths that enter the ocular for each position of the prism.

A perfectly general proof of the following (supposedly new[1]) theorem will now be given. *When a ray of light passes through a triangular prism in a principal plane in such a manner as to suffer one internal reflection and two refractions, and having the angle of emergence equal to the angle of incidence (minimum deviation), there exists one, and only one, axis of rotation which will cause the emergent ray to remain in a fixed position while the prism is turned around this axis and the incident ray is maintained immovable. The axis is the intersection of the face at which total reflection occurs with*

[1] W. E. Forsythe proved a part of this theorem for the special case where $\omega=90^\circ$; *loc. cit.*

the plane bisecting the interior angle between the incidence and emergence faces.

In Fig. 5 $X'O'Y'$ represents a principal section of the prism. In this diagram ($\tau=30°$) $\phi=52°$, $\sigma=82°$, and $\delta=\omega=76°$. The length $\overline{O'Y'}$ will be denoted by b. The broken line $CITED$ indicates the path of the complete ray when the angle of emergence equals the angle of incidence ι. Under this condition, as demonstrated above, the total deviation of the ray will have the constant value ω; hence, since the incident ray $\overline{CI}$ has an invariable position,

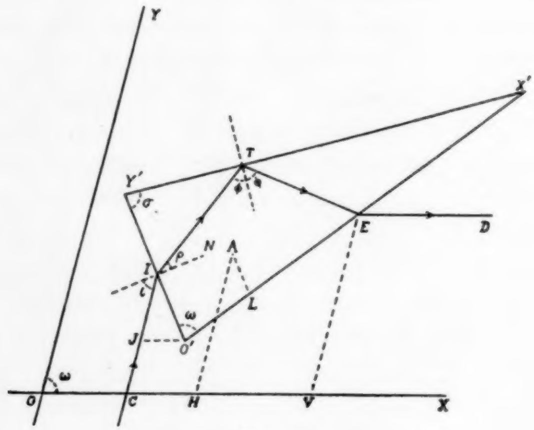

FIG. 5

a set of fixed oblique co-ordinate axes may be taken such that $\overline{OY}$ and $\overline{OX}$ are parallel to the incident and emergent segments ($\overline{CI}$ and $\overline{ED}$) of the ray, respectively. In other words, $\overline{OX}$ and $\overline{OY}$ are parallel respectively to the optic axes of the telescope and collimator. $\angle XOY=\omega$. The point A indicates the axis of rotation which is fixed with respect both to the immovable co-ordinate frame XOY and to the rotating frame $X'O'Y'$. $\overline{OC}=x_c$, $\overline{OH}=x_o$, $\overline{HA}=y_o$, $\overline{VE}=y_\iota$, $\overline{O'L}=x'_o$, and $\overline{LA}=y'_o$. When the prism is rotated around the axis through A (x_c, x_o, y_o, x'_o, y'_o, being kept constant) the point of emergence E will move along the face $\overline{O'X'}$ and, in general, the line $\overline{ED}$ will move normal to OX, so that y_ι will vary as ι changes. The problem is to determine, if possible, one set (or perhaps a locus of sets) of values of x_o, y_o, x'_o, and y'_o such that

y_t will remain constant independently of the changing values of ι (varying colors). Accordingly an analytical expression giving y_t as an explicit function of b, x_c, x_0, y_0, x_0', y_0', σ, ω, and ι will be derived. The condition that y_t shall be independent of ι is expressed mathematically by the vanishing of the algebraic coefficients of the terms containing circular functions of ι as factors.

The work can be conveniently systematized and appreciably simplified by commencing with the co-ordinate system $X'O'Y'$. The abscissa $\overline{O'E}$ of the point of emergence may be obtained by the aid of the equations of the lines $\overline{IT}$, $\overline{X'Y'}$, and $\overline{ET}$, as will now be shown. Let $\overline{O'I}\equiv a$. The angle made by the line $\overline{IT}$ with the axis $\overline{O'X'}$ equals $\omega-(90°-\rho)$ or $\sigma+\frac{3}{2}\omega-180°\equiv\theta$, since $\angle NIT=\rho=\sigma+\frac{1}{2}\omega-90°$, as proved in an earlier paragraph. Hence the point-slope formula $y-y_1=m(x-x_1)$, where $m\equiv\sin\theta/\sin(\omega-\theta)$, leads directly to the equation of $\overline{IT}$, which may be written as

$$x'\sin(\sigma+\tfrac{3}{2}\omega)+(y'-a)\sin(\sigma+\tfrac{1}{2}\omega)=0. \tag{6}$$

The line $\overline{X'Y'}$ has the slope-angle θ equal to $\sigma+\omega$ and it passes through the point $(0, b)$, hence its equation is

$$x'\sin(\sigma+\omega)+(y'-b)\sin\sigma=0. \tag{7}$$

The co-ordinates (x_2', y_2') of the point of total reflection T are obtained by solving the simultaneous equations (6) and (7). By taking advantage of the identity

$$\sin(\sigma+\omega)\sin(\sigma+\tfrac{1}{2}\omega)-\sin\sigma\sin(\sigma+\tfrac{3}{2}\omega)=\sin\tfrac{1}{2}\omega\sin\omega$$

it will be found that

$$\left.\begin{aligned} x_2'&=\frac{(b-a)\sin\sigma\sin(\sigma+\frac{1}{2}\omega)}{\sin\frac{1}{2}\omega\sin\omega}\\ y_2'&=\frac{a\sin(\sigma+\frac{1}{2}\omega)\sin(\sigma+\omega)-b\sin\sigma\sin(\sigma+\frac{3}{2}\omega)}{\sin\frac{1}{2}\omega\sin\omega}\end{aligned}\right\} \tag{8}$$

Next, to obtain the equation of the line $\overline{ET}$. Since, at minimum deviation, $\angle X'ET=90°+\rho=\sigma+\frac{1}{2}\omega$, the equation of $\overline{ET}$ may be written

$$(x'-x_2')\sin(\sigma+\tfrac{1}{2}\omega)+(y'-y_2')\sin(\sigma-\tfrac{1}{2}\omega)=0.$$

Consequently, the abscissa $x_3'(y_3'=0)$ of the point E is given by

$$x_3'=\frac{x_2' \sin(\sigma+\tfrac{1}{2}\omega)+y_2' \sin(\sigma-\tfrac{1}{2}\omega)}{\sin(\sigma+\tfrac{1}{2}\omega)}. \tag{9}$$

Substitution in (9) of the expressions (8) for x_2' and y_2' leads, after suitable reductions, to

$$x_3'=\frac{2b \sin\sigma \cos\tfrac{1}{2}\omega-a\sin(\sigma+\tfrac{1}{2}\omega)}{\sin(\sigma+\tfrac{1}{2}\omega)}. \tag{10}$$

These reductions may be advantageously effected by making use of the following identities:

$$\sin^2(\sigma+\tfrac{1}{2}\omega)-\sin(\sigma+\tfrac{3}{2}\omega)\sin(\sigma-\tfrac{1}{2}\omega)=\sin^2\omega,$$

$$\sin\sigma\sin(\sigma+\tfrac{1}{2}\omega)-\sin(\sigma-\tfrac{1}{2}\omega)\sin(\sigma+\omega)=\sin\tfrac{1}{2}\omega\sin\omega.$$

It is now necessary to change from the co-ordinate frame $X'O'Y'$ to XOY. When the origin O is moved to $O'(x_1, y_1)$, and the new axes are turned through an angle θ, the old ordinate y of any point is connected with the new co-ordinates x' and y' according to the following equation of transformation:

$$y=y_1+[x'\sin\theta+y'\sin(\omega+\theta)]\csc\omega.$$

In the present case $\theta=90^\circ-\iota$, so that, for the point $E(x_3', 0)$,

$$y_e=y_1+x_3'\cos\iota\csc\omega,$$

from which, by (10)

$$y_e=y_1+\frac{[2b\sin\sigma\cos\tfrac{1}{2}\omega-a\sin(\sigma+\tfrac{1}{2}\omega)]\cos\iota}{\sin\omega\sin(\sigma+\tfrac{1}{2}\omega)}.$$

The triangle $O'IJ$ shows that

$$a\cos\iota=(x_1-x_c)\sin\omega,$$

therefore

$$y_e=x_c-x_1+y_1+\frac{b\sin\sigma\cos\iota}{\sin\tfrac{1}{2}\omega\sin(\sigma+\tfrac{1}{2}\omega)}. \tag{11}$$

The co-ordinates of $O'(x_1, y_1)$ will now be derived from the equations of the lines $\overline{O'X'}$ and $\overline{O'Y'}$. The angle which the normal drawn from O to $\overline{O'X'}$ makes with $\overline{OX}$ equals $360^\circ-\iota$. The per-

pendicular distance from $A(x_0, y_0)$ to $O'X'$ equals $-y_0' \sin \omega$, so that, from the standard normal form

$$x \cos \alpha + y \cos (\omega - \alpha) - p = 0$$

the equation of $\overline{O'X'}$ is found to be

$$(x - x_0) \cos \iota + (y - y_0) \cos (\omega + \iota) - y_0' \sin \omega = 0. \tag{12}$$

For the line $\overline{O'Y'}$ $\alpha = \omega - \iota$, and the perpendicular distance from A to $\overline{O'Y'}$ is $x_0' \sin \omega$, hence the equation of $\overline{O'Y'}$ becomes

$$(x - x_0) \cos (\omega - \iota) + (y - y_0) \cos \iota + x_0' \sin \omega = 0. \tag{13}$$

The identity $\cos^2 \iota - \cos (\omega + \iota) \cos (\omega - \iota) = \sin^2 \omega$ facilitates the reduction of the solution of equations (12) and (13) to

$$\left.\begin{aligned} x_1 &= x_0 + \frac{x_0' \cos (\omega + \iota) + y_0' \cos \iota}{\sin \omega} \\ y_1 &= y_0 - \frac{x_0' \cos \iota + y_0' \cos (\omega - \iota)}{\sin \omega} \end{aligned}\right\} \tag{14}$$

Substitution of these expressions for x_1 and y_1 in equation (11) leads to the following final fundamental function:

$$y_t = x_c - x_0 + y_0 + (x_0' - y_0') \sin \iota - \left[x_0' + y_0' - \frac{b \sin \sigma}{\cos \frac{1}{2}\omega \sin (\sigma + \frac{1}{2}\omega)} \right] \cot \tfrac{1}{2}\omega \cos \iota. \tag{15}$$

The position of the emergent ray may be made independent of the angle of incidence by equating to zero the coefficients of $\sin \iota$ and $\cos \iota$. Accordingly

$$\left.\begin{aligned} x_0' - y_0' &= 0 \\ x_0 + y_0' &= \frac{b \sin \sigma}{\cos \frac{1}{2}\omega \sin (\sigma + \frac{1}{2}\omega)} \end{aligned}\right\} \tag{16}$$

Since equations (16) are independent linear functions of x_0' and y_0', and as the processes involved in the derivation of formula (15) are perfectly general, it follows that one, and only one, axis of rotation exists. The first of these equations shows that (x_0', y_0') satisfies the relation $x' - y' = 0$, which represents a plane bisecting the angle between the incidence and emergence faces. To complete the

proof of the theorem under consideration it remains to show that the point (x_0', y_0') lies on the line $\overline{X'Y'}$. By combining equation (7) with $x'-y'=0$ it will be found that

$$x'=y'=\frac{b \sin \sigma}{\sin(\sigma+\omega)+\sin\sigma}=\frac{b\sin\sigma}{2\cos\frac{1}{2}\omega\sin(\sigma+\frac{1}{2}\omega)}\,.$$

But this is precisely the solution of equations (16), consequently the theorem has been demonstrated in its entirety.

Not only is it necessary to know the position of the axis of rotation with respect to the faces of the prism, but the location of this line with reference to the optic axes of the collimator and telescope must also be determined. When equations (16) are satisfied, formula (15) reduces to

$$y_t=x_c-x_0+y_0.$$

Now the incident and emergent rays must coincide respectively with the optic axes of the collimator and telescope, so that, if these axes are to be taken as lines of reference, the co-ordinate axes $\overline{OY}$ and $\overline{OX}$ must be translated until they coincide with $\overline{CI}$ and $\overline{ED}$, in the order named. These two conditions are expressed mathematically by $x_c=y_t=0$. Then $x_0-y_0=0$; hence, *the axis of rotation must lie in a plane which bisects the angle ω between the optic axes of the collimator and telescope.*

The actual value of x_0 and y_0 will depend upon the arbitrary choice of the point on the incidence face at which a ray corresponding to a given index of refraction shall meet this plane. Accordingly let ι_0 be the value of the angle of incidence for the ray that strikes the incidence face at a point which divides b in the ratio $1:r$. That is, $\overline{O'I}:\overline{IY'}=a:(b-a)=1:r$. $a=b/(1+r)$, $x_c=0$, and $\iota=\iota_0$ so that the relation $a\cos\iota=(x_1-x_c)\sin\omega$, found earlier, reduces to

$$x_1=\frac{b\cos\iota_0}{(1+r)\sin\omega}\,.$$

Substituting this value of x_1, together with the common value of x_0' and y_0' given by (16), in the first of formulae (14), it will be found that

$$x_0=y_0=\frac{b}{\sin\omega}\left[\frac{\cos\iota_0}{1+r}-\frac{\sin\sigma\cos(\iota_0+\frac{1}{2}\omega)}{\sin(\sigma+\frac{1}{2}\omega)}\right]. \tag{17}$$

As a numerical example, let $n=1.60$, $r=\frac{4}{3}$, $\sigma=82°$, $\tau=30°$, and $\omega=76°$. Then $\iota_o=53°7'48''$ and $x_o=y_o=0.288b$. A sectional view of the prism in this position is shown in Fig. 6.

The value of x_o from chosen values of r and ι_o ($r=1$, and ι_o corresponding to 5893 A, say) having been determined, equation (17) may be used to find the value of r associated with some other angle of incidence ι_o' (for the C line or the F line, perhaps), as x_o is now a known quantity. Let w and r_o' denote respectively the half-width of the incident beam and the value of r calculated from b, x_o, ι_o', σ, and ω. Then it is easy to show that the inequalities

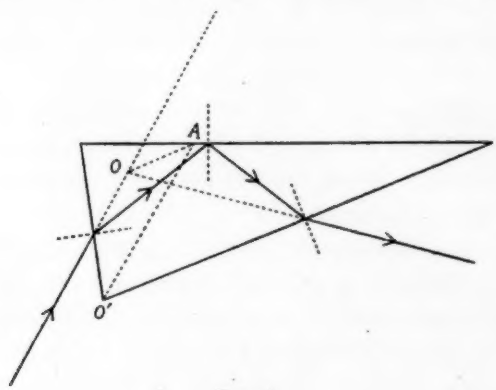

FIG. 6

$w<(b \cos \iota_o')/(1+r_o')$ and $w<(br_o' \cos \iota_o')/(1+r_o')$ must be fulfilled in order that the edge rays may strike the incidence face b. It has been tacitly assumed that r_o' comes out positive, for if it were negative b would be divided *externally* and the axial or chief ray itself would not hit the material segment b of the incidence plane. In any event, special care must be taken to avoid errors in such cases as are typified by Fig. 4 (c).

In conclusion I desire to call attention to the fact that the comments made by W. E. Forsythe, on page 278 of his paper, seem at least to imply that my earlier article did not cover precisely the same problem as the one which he so neatly solved. If this interpretation be correct, then his statements admit of appropriate modification for two reasons: (i) on page 42 of my earlier paper a feasible method for moving the prism which covers all cases was

given, and (ii) formula (18) on page 49 (*loc. cit.*)—which is a special case ($\omega = 90°$) of equation (15) given above—leads at once to his result. It is perfectly true, however, that I did not explicitly deduce the co-ordinates of his axis from formula (18). My second solution was approximate to a sufficiently high degree of accuracy, whereas his solution is mathematically exact.

SUMMARY

1. The important properties possessed by so-called quadrilateral prisms giving a deviation of 90° have been generalized for any feasible constant deviation δ. The most fundamental single fact brought out by this part of the analysis is that, at effective minimum deviation, the deviation is always equal to the interior angle between the incidence and emergence faces of the prism ($\delta = \omega$).

2. The position of the axis of rotation given by W. E. Forsythe[1] for $\omega = 90°$ has been shown to hold for any triangular prism producing a constant deviation δ.

3. It has been demonstrated that the axis of rotation, which must be perpendicular to the plane determined by the optic axes of the collimator and telescope, must lie in the plane which bisects the angle ω between these optic axes.

4. A formula connecting the angle of incidence with the ratio in which the point of incidence of the axial ray divides the incidence face has been derived. As corollaries of this equation, criteria are given for determining whether the extreme rays of a beam will or will not strike the incidence face of the prism.

SLOANE PHYSICAL LABORATORY, YALE UNIVERSITY
November 29, 1917

[1] *Loc. cit.*

THE VISIBILITY OF RADIATION IN THE BLUE END OF THE VISIBLE SPECTRUM[1]

By L. W. HARTMAN[2]

Through the courtesy of its director it was the privilege of the writer in June and July, 1916, to work at the Nela Research Laboratory of the National Lamp Works of General Electric Company, at Nela Park, Cleveland, Ohio. Inasmuch as it did not seem advisable to study the problem originally planned by the writer, Director Hyde suggested that a study be made of the visibility of radiation in the blue end of the spectrum. The results herewith presented were obtained as a consequence of such an investigation.

Among the most thorough as well as the most important determinations of visibility made thus far may be mentioned those of König,[3] Langley,[4] Thürmel,[5] Ives,[6] Bender,[7] Nutting,[8] Hyde and Forsythe,[9] and Coblentz and Emerson.[10] The visibility of radiation for the average eye varies with the wave-length, having the greatest value in the region of 555 $\mu\mu$ and decreasing rapidly to low minimum values at either end of the visible spectrum. The visibility for a given wave-length V_λ may be expressed by the relation

$$V_\lambda = \frac{I_\lambda}{E_\lambda},$$

where I_λ represents for the given wave-length interval the luminous intensity of the light-source measured in light units, and E_λ is the

[1] Communicated from Nela Research Laboratory, National Lamp Works of General Electric Company, Nela Park, Cleveland, Ohio.

[2] University of Nevada.

[3] *Ges. Abhandlungen.*

[4] *American Journal of Science*, **36**, 359, 1888.

[5] *Annalen der Physik*, **33**, 1139, 1910.

[6] *Philosophical Magazine* (6), **24**, 853, 1912.

[7] *Annalen der Physik* (4), **45**, 105, 1914.

[8] *Philosophical Magazine* (6), **29**, 301, 1915.

[9] *Astrophysical Journal*, **42**, 285, 1915.

[10] *Bulletin of the Bureau of Standards*, **14**, 167, 1917.

radiation from the source for the same interval of wave-length measured in energy units. The latter quantity can be computed from Wien's equation and the color-temperature of the source, after which corrections for dispersion, stray light, and absorption of the optical system must be made. When values of I_λ have been determined, V_λ can be computed.

Two methods have been used hitherto to obtain the value of I_λ in this ratio. The first, as in the ordinary photometer based on equality of brightness, involves a direct comparison of the illumination produced by light of successive wave-lengths in the visible spectrum with that produced by a second source considered as a standard; and the second involves the use of the flicker photometer, in which the disappearance of flicker is the criterion of equality. In either case the light from the comparison source may be kept constant in color, or the step-by-step method may be employed, in which case the color of the comparison source is changed at those points where the color-differences exceed a predetermined amount.

The direct-comparison method was used by König, Langley, and Hyde and Forsythe, while Ives, Bender, Thürmel, Nutting, and Coblentz and Emerson have used the flicker method. Although much work has been done on the subject, there is doubt that these two methods give the same results for very great color-differences; indeed, it has been shown that in certain cases they do not.[1] The measurements of Coblentz and Emerson[2] have extended from 400 $\mu\mu$ to 750 $\mu\mu$, while those of Hyde and Forsythe,[3] who carried their observations into the red end of the spectrum farther than anyone else, extended from 620 $\mu\mu$ to 770 $\mu\mu$.

The great difficulty in the way of determining the visibility of radiation far out in the red or the blue end of the spectrum is the small amount of light available. When it is realized that the visibility of radiation for the average eye in going from the position of maximum sensibility to about 400 $\mu\mu$ varies by a very large factor, perhaps 40,000 to 60,000, as may be seen by considering the result of others in conjunction with those given below, it will be evident

[1] Luckiesh, *Electrical World*, **67**, 621, 1913; *Physical Review* (2), **4**, 1, 1914.

[2] *Bulletin of the Bureau of Standards*, **14**, 167, 1917.

[3] *Astrophysical Journal*, **42**, 285, 1915.

that a source that would be very luminous, considered as a whole, would be quite weak if only a small interval of wave-length were taken in the deep blue.

The method utilized here was one that previously had been used in another investigation[1] in this laboratory. It consisted of an adaptation of the arrangement of the parts of the Holborn-Kurlbaum optical pyrometer. One advantage of this method is that it permits the use of greater brightness so that measurements may be made in the extreme blue region of the spectrum. The arrangement of the parts of the apparatus is shown in Fig. 1. By means of a projection lens at *B*, the image of the source of light, *A*, a broad, vertical, incandescent filament of a gas-filled tungsten

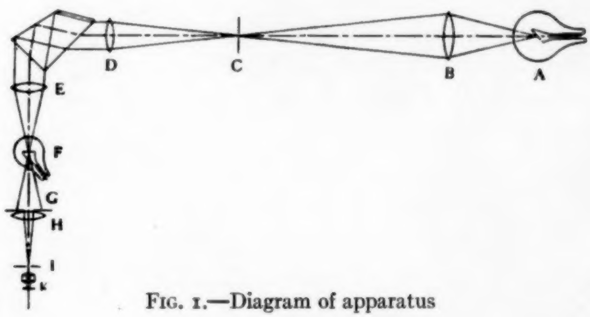

FIG. 1.—Diagram of apparatus

lamp, was thrown upon the collimator slit, *C*, of a Hilger constant-deviation spectrometer. After the light from this source had passed through the prism the spectrum was formed in the focal plane of the telescope of the spectrometer, where the horizontal filament of a small tungsten lamp, *F*, serving as a comparison lamp, was mounted. The lens, *H*, projected an image of the incandescent filament of this lamp, together with the spectrum of the source, *A*, on a narrow adjustable vertical slit, *I*, placed in the focal plane of the eyepiece, *K*. In making the observations with the different observers the width of the collimator slit was maintained at a constant width of 1.00 mm, and the slit in the eyepiece was kept at a constant width of 0.2 mm. Upon turning the drum attached to the prism-stand of the spectrometer, any desired portion of the

[1] *Ibid.*

spectrum of the source *A* could be brought into the field of view under the eyepiece slit of the telescope, and could be compared in brightness with the incandescent filament of the pyrometer lamp, which was supplied with constant current and therefore was maintained at a constant temperature. Throughout the whole series of observations the brightness of the source of light (*A*) was constant, as the current through it was kept at 13.54 amperes, giving a color-temperature[1] of 2695° K. This source possessed two distinct advantages: it was broad, the filament being 1.8 mm wide, and as the projection lens gave a magnification of 1.7 the image more than covered the collimator slit, the cone of rays still being large enough to fill the collimator lens, and the image was exceedingly bright. In order to eliminate as far as possible the effect of stray light and at the same time to reduce the color-difference between the incandescent filament and the particular portion of the spectrum under observation, a piece of blue glass of known spectral transmission was mounted in the eyepiece of the telescope. The image formed by the eyepiece was approximately 1.00 mm in diameter.

The value of the current in the filament of the comparison lamp (*F*) was set so as to match the brightness of a region far out in the blue end of the spectrum of the source (*A*); and then the current in the filament, and therefore the brightness of the filament, was maintained constant; and the different regions of the spectrum of the source (*A*) were compared in brightness with it. This was accomplished by introducing, immediately in front of the collimator slit, rotating sectored disks which reduced by a definite, known amount the apparent brightness of the source, and then finding the position of the wave-length drum of the spectrometer, which gave apparent equality between the two. The wave-length drum of the spectrometer had been calibrated by reference to known spectral lines, and the constancy of the calibration was frequently checked. Before making the final observations, trial tests were made with three different widths of the collimator slit, viz., 0.5 mm, 1.00 mm, and 1.50 mm, and with three different currents, and therefore, upon plotting the luminosity-curves, it was found that if they were arbitrarily made to agree at some one wave-length they then agreed

[1] Hyde, Cady, and Forsythe, *Physical Review* (2), **10**, 395, 1917.

within the limits of errors throughout the overlapping wave-lengths. Since the variations in brightness for the average eye for these curves were of the order of 40 to 1 (a ratio of 53 to 1 for the individual whose curves are given later in Fig. 4), it is believed that the possibility of an error due to the Purkinje effect has been eliminated. In addition it may be noted that a small field was used. As noted heretofore, throughout this series of observations the widths of the slits of the collimator and eyepiece were maintained constant, namely, at 1.00 mm and 0.20 mm, respectively.

The energy-curve of the light-source (A) was determined by comparison with a black body. Using the temperature thus obtained, 2695° K, the distribution of energy was calculated from Wien's equation, where the value of C_2 was taken equal to 14,350 micron degrees. These values were then corrected for dispersion and selective absorption of the prism-and-lens system, and for the absorption of the pyrometer lamp, and the luminosity was corrected for width of slits, for scattered light, and for the absorption of the blue glass in the eyepiece of the telescope. The corrections for the selective absorption of the prism-and-lens system of the instrument were based on determinations made by Dr. Forsythe for this same apparatus and used in the paper mentioned above.[1] In these determinations the transmission of the various parts of the optical system for the different wave-lengths was obtained from measurements of brightness with a spectral pyrometer. In case of the prism the dispersion effect was overcome by using an extended source that was fairly uniform.

The following method was used to determine the brightness of the scattered light: Inasmuch as the field of view of the spectrum in the eyepiece was limited in height by a diaphragm placed in front of the collimator slit (C), the filament of the comparison lamp (F) was moved so as to be about a millimeter above or below the field of the spectrum of the source (A). In this position the current in the comparison lamp was adjusted so that its brightness matched the brightness of the scattered light alone, as seen through the blue glass. The comparison lamp was then replaced in its proper position at the center of the field of the spectrum, as seen in the

[1] Hyde and Forsythe, *Astrophysical Journal*, **42**, 285, 1915.

eyepiece, and the light from the source (A) was reduced, by means of rotating sectors placed immediately in front of the collimator slit

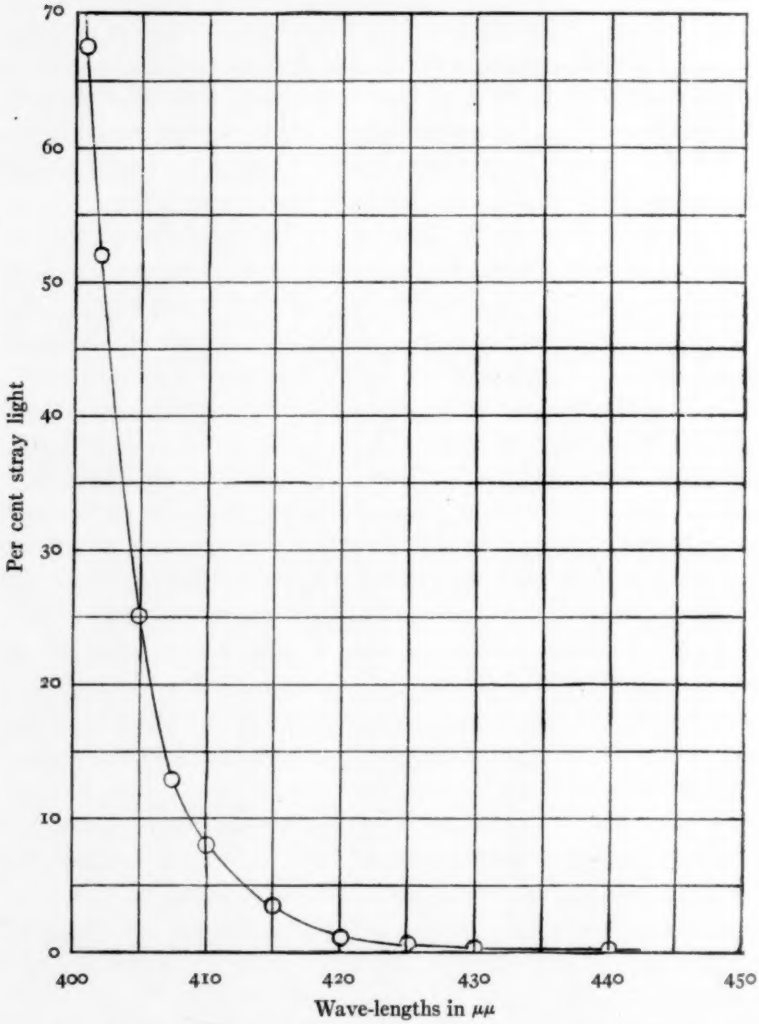

FIG. 2.—Curve showing percentage variation of stray light through blue glass, with wave-length of transmitted light.

(C), to a value such that the brightness of the comparison lamp, with still the same current flowing through it, matched the bright-

ness of the direct radiation plus the scattered light. By varying the current through the filament of the comparison lamp the comparison could be made for any desired wave-length. Knowing the fraction of the total light transmitted through the rotating sector, the ratio of the scattered light to the total radiation could be easily

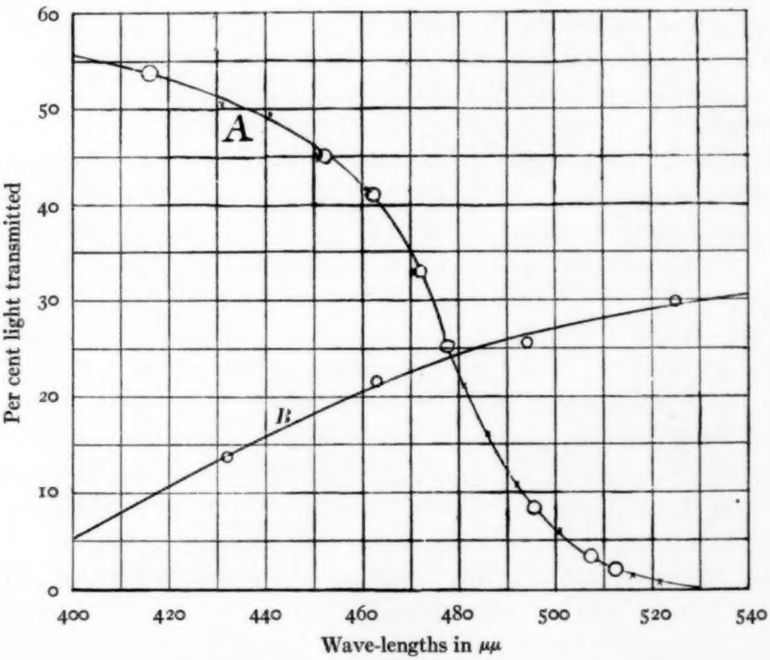

FIG. 3.—Transmission curves: *A*, Blue glass; *B*, Prism and lenses

computed. The results of these determinations are shown graphically in Fig. 2. That this method gives a satisfactory determination of the percentage transmission of the scattered light has been shown elsewhere.[1] The transmission of the blue glass in the eyepiece of the telescope was also determined by Dr. Forsythe, and the results are shown graphically in Fig. 3*A*. The transmission-curves of the other parts of the optical system of the apparatus are shown in Fig. 3*B*.

[1] Hyde and Forsythe, *Astrophysical Journal*, **42**, 289, 1915.

In this series of visibility observations, measurements were made by twenty different individuals, all of whom were trained laboratory observers. The final results are tabulated in Table I. In this table the results have been reduced to a common value at $\lambda = 450\ \mu\mu$. The settings of the spectrometer by each individual were made with at least three different values of current in the pyrometer lamp, and with from six to eight different rotating sectors, the calibration of which had been determined with great accuracy. Frequent check readings were also made by the individual observers. After the readings by an individual observer had been recorded, curves were plotted with wave-length for abscissae, and logarithms of luminosities for ordinates. In these plots the luminosities for the three different degrees of brightness of the pyrometer lamp were made equal in the portions of the spectrum where they overlapped. To do this a number of readings of the differences in the ordinates of the two curves of greater luminosities were taken, and an average was calculated by which the adjustments were made. Thus a smooth curve was obtained, extending from the shortest wave-length at which observations were made to the longest wave-length at which observations were made, and the values for each observer given in Table I are based on the readings from such a smooth curve. A set of such curves, together with the resultant curve, is shown in Fig. 4. In this set of curves, logarithms of luminosities are plotted as ordinates and wave-lengths as abscissae. The three different values of current used throughout in the pyrometer lamp gave values of brightness corresponding to color-match temperatures of 1312° K, 1424° K, and 1568° K, respectively, which in turn gave through the blue-glass filter brightnesses of 0.000033, 0.00019, and 0.0013 candles per square centimeter, respectively.

The relative values of luminosity for radiation for the different observers for a given wave-length in the blue end of the spectrum compared with the average luminosity of a black body at 1315° K through the blue-glass filter used are shown in Table II. In obtaining these data both the source and the comparison lamp were kept constant as in the regular determinations, the former being at 2695° K and the latter at 1312° K, as settings were made by each observer. Data were thus obtained as above, from which curves

TABLE I

VISIBILITY DATA ON TWENTY SUBJECTS IN THE BLUE END OF THE SPECTRUM

Wave-Length	L.W.H.	W.E.F.	E.O.H.	P.W.C.	W.W.	L.T.T.	F.E.C.	J.K.	C.F.S.	A.G.W.	C.F.L.	I.H.V.	C.F.K.	J.H.M.	W.E.B.	R.G.B.	H.C.M.	G.P.L.	H.M.J.	E.P.H.
410	1.15	2.15	1.40	2.50	1.40	2.15	1.90	2.95	2.60	1.65	0.95	0.85	1.05	1.50	2.65	1.55	1.55	1.40	1.40	2.15
420	9.0	13.0	10.0	13.5	9.6	13.5	15.0	15.5	14.5	10.0	10.5	8.9	7.3	9.0	11.9	11.0	11.1	10.0	9.8	15.5
430	29.0	32.0	39.5	30.5	33.5	35.0	37.5	37.0	37.5	36.5	28.0	23.0	27.0	30.0	30.0	28.0	35.0	32.0	33.5	38.0
440	60.	58.	58.	63.	62.	62.	71.	64.	64.	71.	56.	54.	57.	61.	59.	56.	66.	61.	63.	66.
450	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
460	155	160	170	150	160	170	125	155	160	145	165	160	155	150	155	160	145	150	150	125
470	220	265	285	220	245	265	155	250	270	235	265	255	240	220	250	250	245	215	265	190
480	315	425	480	340	395	420	205	355	485	315	450	410	385	350	400	415	385	255	485	255
490	470	725	770	530	550	670	295	510	850	705	755	600	581	545	695	770	570	535	895	380
500	650	1135	1075	755	590	1015	415	690	920	1250	1080	760	750	835	1110	1280	780	780	1640	590

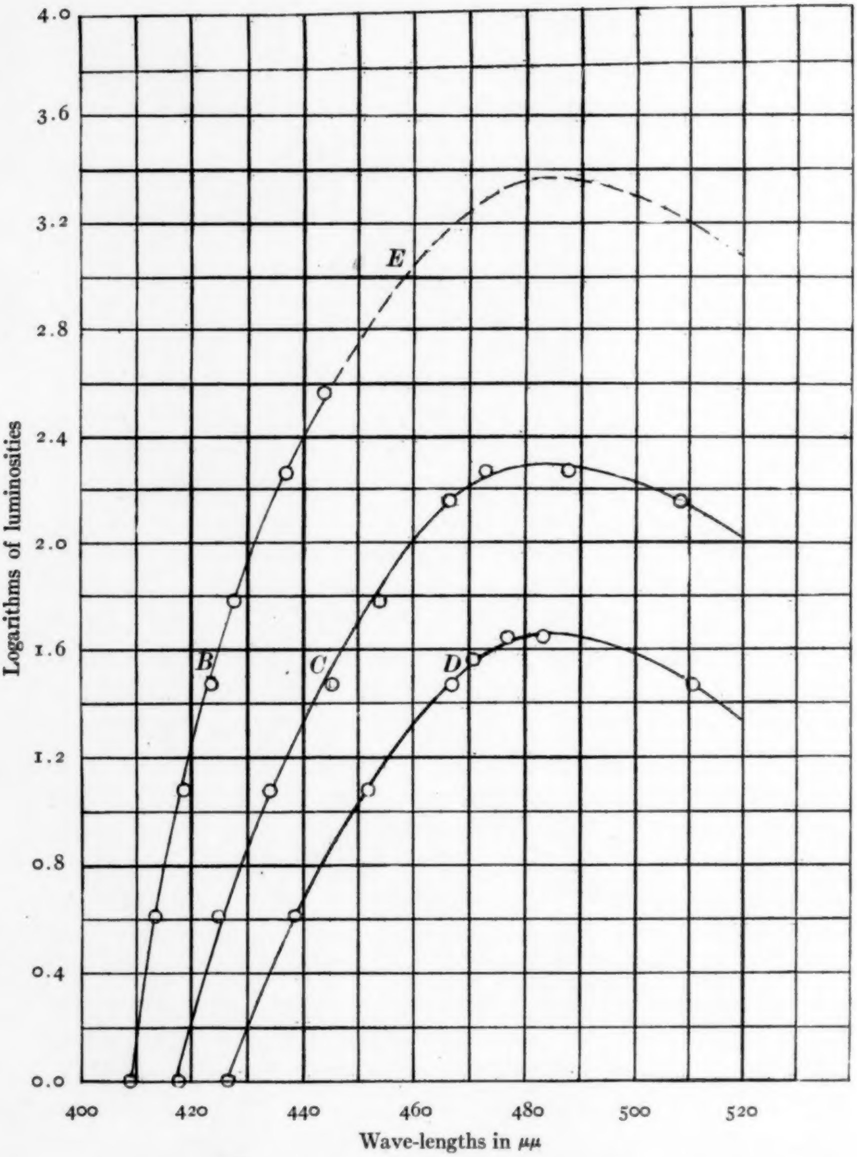

FIG. 4.—Curves showing the relation between logarithms of luminosities and wave-lengths with three different values of current through the comparison lamp. In curve *B* the current through the lamp (*F*) was 0.250 amperes; in curve *C*, 0.285 amperes; and in curve *D*, 0.325 amperes. The average differences between the ordinates of curves *B* and *C* and curves *C* and *D* for the wave-length interval where overlapping occurs were determined and the luminosities of *C* and *D* were made equal in this region to the luminosity of *B* for the given wave-length intervals. The curve *E*, represented by the broken line, gives the extended portion of the curve *B*, based on this adjustment of the luminosity values of curves *C* and *D*. The circles on curve *B* represent the observations made with the lowest current value in the lamp (*F*).

were plotted. From these curves one can determine the relative values of the luminosity for radiation for the different observers for light of the desired wave-length as already mentioned.

TABLE II

RELATIVE VALUE OF THE LUMINOSITY OF RADIATION FOR TWENTY DIFFERENT OBSERVERS FOR LIGHT AT $\lambda = 430\ \mu\mu$ COMPARED WITH THE AVERAGE LUMINOSITY OF A BLACK BODY AT 1312° K THROUGH THE BLUE-GLASS FILTER USED

H. M. J	58	E. O. H	109
W. E. B	69	G. P. L	111
R. G. B	69	I. H. V	115
J. H. M	70	P. W. C	115
C. F. K	85	E. P. H	120
A. G. W	87	W. W	120
H. C. M	100	C. F. S	126
W. E. F	101	J. K	145
L. T. T	102	L. W. H	162
C. F. L	107	F. E. C	166

The average values of visibility for radiation of the twenty observers, together with the values obtained from the data of Nutting and of Coblentz and Emerson, are given in Table III, in

TABLE III

Wave-Lengths	Mean Visibility of Twenty Subjects	Mean Values Given by Nutting*	Mean Values Given by Coblentz and Emerson
$\mu\mu$			
410	1.7	9.5	24
420	11.4	17.1	42
430	32.6	30.3	59
440	61.6	58.0	71
450	100	100	100
460	153	168	137
470	240	266	202
480	376	392	305
490	620	566	474
500	905	828	770

* These values, kindly furnished by Dr. Nutting, differ slightly from his published data, owing to a redetermination of the distribution of energy in the spectrum of the acetylene flame used.

which all the observations have been reduced to 100 for $\lambda = 450\ \mu\mu$. It will be noted that the values obtained by the method outlined in this paper are lower for the extreme blue than the values given in the two comparison columns. Values for 400 $\mu\mu$ were obtained

by extrapolation from 405 $\mu\mu$ or 407 $\mu\mu$, but it was decided to include in the table only those values obtained from data based on actual observations in the region of the extreme short wave-lengths.

One can test the accuracy of the values of visibility of radiation for the wave-lengths indicated in Table III by computing for some definite temperature-interval the effective wave-length[1] of the blue glass mounted in the eyepiece of the telescope and comparing this with the value determined experimentally. Using the relation

$$\lambda_e = \frac{C_2 \log (e)}{\log (\phi_2/\phi_1)} \cdot (1/T_1 - 1/T_2) ,$$

the computed value of the effective wave-length for the temperature interval 1781° K to 2475° K, for two thicknesses of the glass used, was found to be 466.8 $\mu\mu$, while the value of the effective wave-length for the same two thicknesses of blue glass, determined experimentally by Dr. Forsythe, was found to be 467 $\mu\mu$. In the equation written above, λ_e is the effective wave-length of the blue glass, and C_2 equals 14,350 micron degrees,

$$\phi = \int_0^\infty E_\lambda \, V_\lambda \, t_\lambda ,$$

E_λ being the energy for the given wave-length interval computed from Wien's equation, V_λ the visibility value given in Table III, t_λ the transmission for the given wave-length of the blue glass used in this investigation, and T_1 and T_2 the two temperatures on the Kelvin scale for which ϕ_1 and ϕ_2, the corresponding luminous fluxes through the blue glass, were obtained. This seems to be a valuable check on the accuracy of the results obtained in the present work.

SUMMARY

1. With the aid of a suitable blue-glass screen used in connection with the direct-comparison method of determining the visibility of radiation in the blue end of the spectrum, measurements have been made by twenty different observers, and values of visibility between the limits 410 $\mu\mu$ and 500 $\mu\mu$ have been obtained.

[1] Hyde, Cady, and Forsythe, *Astrophysical Journal*, **42**, 294, 1915.

Corrections for slit-widths, for scattered light, for the absorption of the optical system of the apparatus, have been made. Inasmuch as a very bright source of light was employed, and as a bright but small retinal image was used, and as consistent results with no apparent shift of the visibility-curves were obtained with different widths of the collimator slit and with different brightnesses of the comparison source, it is believed that any error due to the Purkinje effect has been eliminated.

2. Using the values of visibility of radiation obtained by this method for the interval of wave-length mentioned above, and for the interval of temperature 1781° K to 2475° K, the effective wave-length for two thicknesses of the blue-glass filter has been found to be 466.8 $\mu\mu$. The experimentally determined value of the effective wave-length is 467 $\mu\mu$.

In conclusion the writer wishes to express his thanks to the various observers for their time, patience, and unfailing courtesy in making the observations, and especially to Dr. W. E. Forsythe and Dr. A. G. Worthing for many helpful suggestions, for the determination of numerous constants involved in this investigation, and for the free use of many constants and computations used in this paper. To the director, Dr. E. P. Hyde, and to Mr. F. E. Cady the writer wishes to acknowledge his indebtedness for the generous treatment accorded him in the loan of apparatus and the full privileges of the Research Laboratory.

NELA RESEARCH LABORATORY
NATIONAL LAMP WORKS OF GENERAL ELECTRIC COMPANY
NELA PARK, CLEVELAND, OHIO
February 1918

NOTE ON THE SPECTRUM OF THE ISOTOPES OF LEAD

By LESTER ARONBERG

The spectra of isotopes were first investigated by Russell and Rossi[1] and Exner and Haschek,[2] who examined the spectra of thorium and ionium preparation. But no difference in the lines of the two spectra was found.

Aston[3] has examined the spectrum of the two isotopes of neon obtained by fractional diffusion, but found them to be identical.

More recently Soddy and Hyman,[4] Richard and Lembert,[5] Honingschmid and St. Horovitz,[6] and Merton[7] have compared the spectrum of lead of radioactive origin with that of ordinary lead. Soddy and Hyman, who worked with lead from thorite, found that the line λ 4760.1 was stronger in ordinary lead than in the thorite lead, but that the spectra in every other respect seemed to be identical. Richard and Lembert and Honingschmid and St. Horovitz also found that the spectra were identical.

Merton, who used a spectrograph of a higher dispersion than those used by the former investigators (about 10 A per mm), measured the principal lead lines between λ 3500 and λ 4100 in order to detect any difference in wave-length, but he found that no differences greater than 0.03 A (which was within the limit of experimental error) occur in the lines of the two different spectra. According to Professor Hicks's theory, the atomic-weight term enters exactly into the separation of doublets and triplets in series spectra. Assuming that lead has a doublet series spectrum with a separation of 50 A at λ 4000, it was calculated by J. W. Nicholson that if the radio-lead used by Merton was 0.5 unit less

[1] *Proceedings of the Royal Society of London*, **87**, 478, 1912.

[2] *Sitzungsberichte der k. Akad. Wiss., Wien* (II*a*), **121**, 175, 1912.

[3] British Association Meeting 1913, p. 403.

[4] *Chemical Society Transactions*, **105**, 1402, 1914.

[5] *Journal of American Chemical Society*, **36**, 1329, 1914.

[6] *Sitzungsberichte der k. Akad. Wiss., Wien* (II*a*), **123**, December 1914.

[7] *Proceedings of the Royal Society of London*, **91**, 198, 1914.

than ordinary lead, a shift of the order of 0.15 A should result. But, as stated above, no change in wave-length of this order occurs.

Merton also made a special examination of the line λ 4058 by means of a Fabry and Perot étalon, and found that there is no difference in wave-length as great as 0.003 A (which was within the limit of experimental error) for the line λ 4058 in the spectrum of ordinary lead and of lead from pitchblende.

The present investigation was undertaken at the suggestion of Professor W. D. Harkins, who wished to determine whether the electronic periods are wholly dependent upon the net charge of the nucleus of the atom. It was thought possible that the mass of the nucleus might have a small additional effect. The lead which was used was very kindly supplied by Professor T. W. Richards, who had determined the atomic weight as 206.318. This lead was obtained from Australian carnotite.

In order to obtain a bright source and at the same time to have the spectral lines very sharp and narrow (which is very necessary for work in the higher orders), a slightly modified form of the oxy-cathode arc *in vacuo*, employed by Wali-Mohammad,[1] was used, as shown in Fig. 1. This source had the additional advantage that the radio-lead, of which only about 3 grams were available, was not wasted, for the deposit on the walls of the tube could be dissolved and the solution saved.

Before proceeding with the grating spectrum a comparison spectrum of the two leads was taken on a Hilger quartz spectrograph, which has a dispersion of about 25 A per mm. The two leads were placed in two different vacuum arc lamps. No change of intensity or in number of lines, however, could be observed. A photograph of the two leads—the ordinary lead at a pressure of 0.04 mm and the radio-lead at a pressure of 0.07 mm—is given in Plate V, Fig. 2 (see plate facing p. 102). The hydrogen and nitrogen lines which appear stronger in one than in the other are due to the 0.03 mm pressure between the two arc lamps.

It seems interesting, however, to note here that at first a change of intensity in the lines λ 2833 and 2823 was observed; viz., in the ordinary lead the line 2833 had an intensity of 10 (arbitrary scale)

[1] *Astrophysical Journal*, **39**, 189, 1914.

and 2823 had an intensity of 5, while in the radio-lead the line 2833 was of intensity 5 and the line 2823 of intensity 10. But, after many repeated trials to find out whether it was a true change or not, it turned out that the pressure in the lamp containing the radio-lead was increased after the arc started from 0.05 mm to 0.25 mm. And it is that small difference in pressure which caused the change in intensity. This was verified by changing the pressure in the arc containing the ordinary lead to 0.25 mm, when the same change of intensity was observed, no matter whether the residual gas was hydrogen or air or whether it was ordinary lead or radio-lead.

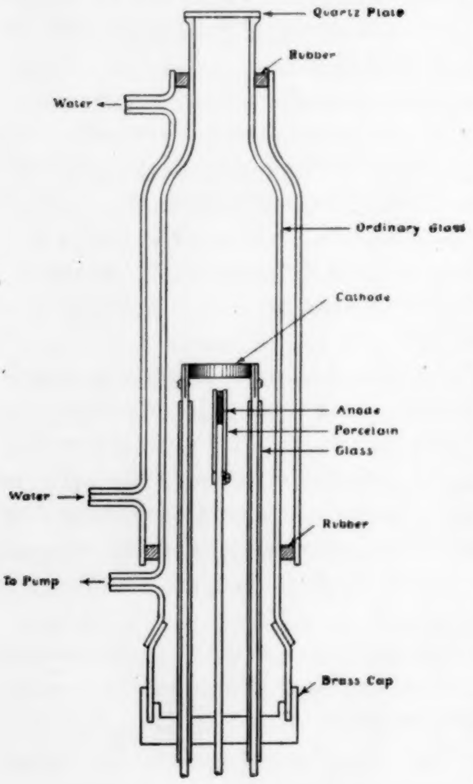

FIG. 1

The sixth order of a Michelson ten-inch plane grating in a Littrow mounting of 30 feet focus was tried next. But, unfortunately, the only line that could be photographed in the higher orders within reasonable exposure times was the line λ 4058, the strongest line of the lead spectrum.

The structure of λ 4058 was studied by Janicki[1] and Wali-Mohammad,[2] using an echelon and the same kind of a light-source as used in this experiment. They found it to possess two satellites, +0.032 and −0.041. In the present investigation it was found that both the ordinary lead and the radio-lead possess only one

[1] *Annalen der Physik*, **29**, 833, 1909. [2] *Astrophysical Journal*, **39**, 189, 1914.

component at −0.0480 and −0.0501 for the radio- and ordinary lead, respectively, the difference being within the experimental error of measuring on account of the faintness of the satellite.

In order to detect any small changes in wave-length, the two arc lamps were connected in parallel to the mercury pump so as to keep the pressure in both identical, about 0.04 mm of mercury, and exposed at the same time to avoid any mechanical shifts. The light from one went through a right-angled prism attached to the slit, while the light from the other lamp passed straight through the slit, thus forming a spectrum the middle of which belongs to one lamp, while the lines above and below belong to the other one.

The voltage, about 40 v., was kept the same in both arcs within one volt (except in a few exposures where one lamp was two or three volts below the other). The current of about 1.1 amp. was kept the same within 0.05 amp. Seven exposures were taken in such a position that the light from the lamp containing the radio-lead passed straight through, while that of the other went through the right-angled prism.

The plates were measured on a Gaertner comparator. The instrument reads directly to thousandths of a millimeter and may be estimated to ten-thousandths. Table I gives the differences in millimeters and angstroms of λ 4058 in the two kinds of lead, the radio-lead having the larger wave-length.

TABLE I

No. of Exposure	Difference in mm	Difference in A
1	0.013	+0.0046
2	.013	+ .0046
3	.019	+ .0068
4	.010	+ .0036
5	.016	+ .0057
6	.012	+ .0043
7	0.010	+0.0036

Then the positions of the lamps were interchanged and six more exposures taken. The measurement gave the values shown in Table II.

The lamps were placed again in their original position and three more exposures were taken, which gave the results shown in Table III.

The average of the foregoing differences in wave-length is 0.012 mm, corresponding to an increase in λ of 0.0043 A for the radio-lead, for the dispersion at λ 4058 in the sixth order = 3.59 A per cm.

TABLE II

No. of Exposure	Difference in mm	Difference in A
8	0.010	+0.0036
9	.010	+ .0036
10	.011	+ .0039
11	.013	+ .0046
12	.016	+ .0057
13	0.010	+0.0036

The satellite was measured on the first four exposures and found to have the same magnitude of change in wave-length. But no accurate measurements could be made on account of the faintness of the component, when the main line was properly exposed.

TABLE III

No. of Exposure	Difference in mm	Difference in A
14	0.013	+0.0046
15	.010	+ .0036
16	0.013	+0.0046

The pressure in the arc must not exceed 0.1 mm of mercury, for if the pressure was higher than 0.1 mm the line broadened and masked the effect.

In order to be certain that the change in wave-length was not due to slight changes in voltage or amperage, the voltage of the ordinary lead was lowered intentionally by 3 volts in Nos. 6 and 7; for by lowering the heating current through the cathode the voltage is changed owing to a smaller electronic emission, while the amperage of the arc can be still kept the same. Nos. 12 and 13 represent the case where the voltage of the radio-lead was lowered by 3 volts. No. 15 was taken when the amperage of the radio-lead was lowered by 0.2 amp., and the voltage was kept the same by increasing the current through the cathode. From the values of the above-mentioned exposures it is seen that slight changes in voltage or amperage have no effect on the shift of the line.

Finally, to establish the reality of the change in wave-length more firmly in a new manner, the radio-lead was removed and replaced by ordinary lead at the same distance from the cathode. The lamps were placed in the same position as that used for the last three exposures. But no difference greater than the experimental error of measuring was found. Four exposures were taken and gave the following values:

No. of Exposure	Difference in mm
1	+.001
2	−.002
3	−.003
4	+.0015

Nos. 3 and 4 were taken by lowering the voltage by 3 volts and the amperage by 0.2 amp., respectively, of the lamp containing the fresh ordinary lead.

The effect of the change of mass of the nucleus on series spectra of helium and hydrogen was studied by Evans[1] and by Paschen.[2] They found it to agree, within the experimental error, with the general theoretical formula deduced by Bohr, viz.:

$$V=\frac{2\pi^2e^2E^2Mm}{h^3(M+m)}\left\{\frac{1}{n_1^2}-\frac{1}{n_2^2}\right\},$$

where e and m are the charge and mass of the electron, E and M the charge and mass of the nucleus, and h is Planck's constant.

Assuming the ordinary lead to have an atomic weight of 207 and the radio-lead 206, E, the nuclear charge, being the same, it is calculated from Bohr's formula that λ 4058 of the radio-lead should be increased by 0.00005 angstrom, which agrees in sign but not in magnitude with the observed value.

The fact that there was an actual shift of the line in the ordinary lead and not a mere broadening of the line would suggest the conclusion that ordinary lead is not a mixture of thorium and radium lead, but is a pure isotope by itself.

In conclusion I wish to thank Professor H. G. Gale for his constant advice and suggestions in the investigation of this problem.

Ryerson Physical Laboratory, University of Chicago
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[1] *Philosophical Magazine*, **29**, 284, 1915. [2] *Annalen der Physik*, **50**, 207, 1916.

THE STRUCTURE OF THE BISMUTH LINE λ 4722

By LESTER ARONBERG

The structure of the bismuth line λ 4722 has been studied by von Baeyer,[1] von Baeyer and Gehrcke,[2] Takamine,[3] Lunelund,[4] and Wali-Mohammad.[5] The former three investigators have used the method of crossed spectra to obtain a very high resolving power, while the latter two have used an echelon grating of very great resolving power.

The results of Lunelund and Wali-Mohammad differ very greatly from those of the former three investigators as to the position of the satellites of the line. Professor H. G. Gale suggested clearing up the question by using a Michelson ten-inch plane grating in a Littrow mounting of 30 feet focus in the sixth order to examine the structure of the line.

TABLE I

Gehrcke and v. Baeyer	v. Baeyer	Takamine	Lunelund	Wali-Mohammad	Grating
+0.316	+0.318	+0.320	−0.031(=+0.314)	−0.029(=+0.316)	+0.318
+ .289	+ .283	+ .284	− .062(=+0.283)	− .061(=+0.284)	+ .284
+ .242	+ .242	+ .238	− .105(=+0.240)	− .103(=+0.242)	+ .240
+ .104	+ .100	+ .102	+ .103	+ .102	+ .102
+ .057	+ .056	+ .056	+ .059	+ .057	+ .056
0.000	0.000	0.000	.000	.000	0.000
....	....	....	.144?	−0.144?	....
....	....	....	0.166?	....	....

As a source of light an oxy-cathode arc was employed, in form slightly modified from that used by Wali-Mohammad (*loc. cit.*). Plate V, Fig. 1, is a photograph of the line taken with a current of 1.5 amp. A number of plates were taken and measured, the results as compared with those of the other investigators being given in Table I.

[1] *Verhandlungen der deutsch. phys. Gesells.*, **9**, 84, 1907.

[2] *Annalen der Physik*, **20**, 285, 1906.

[3] *Tokyo Sugaka-Buturigakkwai Kizi*, 2 ser. VIII, February 1915.

[4] *Annalen der Physik*, **34**, 505, 1911.

[5] *Astrophysical Journal*, **39**, 189, 1914.

PLATE V

FIG. 1.—STRUCTURE OF Bi $\lambda 4722$

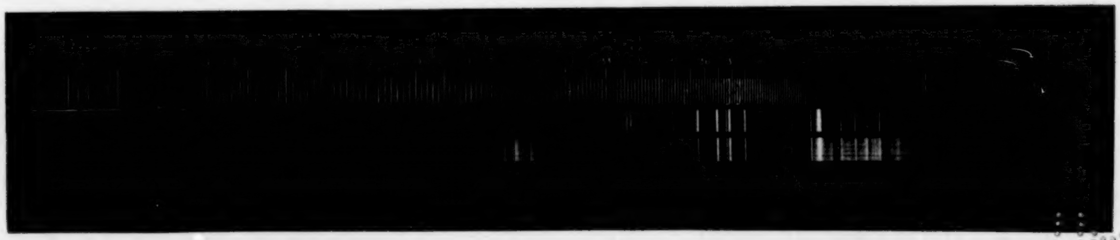

FIG. 2.—SPECTRUM OF LEAD (*a*) AND RADIO-LEAD (*b*)

From this table and from the accompanying photograph it is seen conclusively that the results of Lunelund and Wali-Mohammad are incorrect as to the actual position of the satellites of the line above.

In order to find the true wave-length of the strongest line of the group, as well as of each of its components, a trace of Zn was mixed with the bismuth metal used as the anode. Thus the Zn line λ 4722.164, which is very sharp and single, was photographed at the same time with the Bi line. In a second attempt an equal amount of Zn was mixed with the bismuth, and then the Zn line came out self-reversed, since the current of 1.2 amp. necessary to bring out the Bi line reverses the Zn line. Thus good measurements could be obtained. The true wave-length of each of the components in international units, starting with the strongest one, is as follows:

4722.379, 4722.435, 4722.481, 4722.619, 4722.663, 4722.697.

Ryerson Physical Laboratory, University of Chicago
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ON PARALLAXES AND MOTION OF THE BRIGHTER GALACTIC HELIUM STARS BETWEEN GALACTIC LONGITUDES 150° AND 216°[1]

By J. C. KAPTEYN[2]

1. INTRODUCTION

In *Mount Wilson Contribution* No. 82 an attempt was made to derive the individual parallaxes and motions of practically all the known helium stars between galactic latitudes $\pm 30°$ and galactic longitudes 216°–360°. The present paper aims at a similar determination for stars within the same limits of latitude and between longitudes 150°–216°.[3]

As in the earlier publication, the limits in latitude were thus chosen in order that the overwhelming majority of the B stars should be included in such a way that a tolerable idea of their distribution in space could be obtained by projection on a single plane, namely, that of the galaxy. The limit at $l=216°$ forms the boundary between the stars of *Mount Wilson Contribution* No. 82 and those now under consideration; that at $l=150°$ has been chosen rather arbitrarily. There is at this point some indication of a natural limit in the arrangement of the bright helium stars of small proper motion ($\mu=0\overset{''}{.}016$) shown by Plate IX in *Mount Wilson Contribution* No. 82. Whether later studies confirm this indication or not does not matter much, because it is my intention later to extend the study of the distribution of the helium stars to the remaining galactic longitudes, at least as far as possible. Down to longitude 110° provisional investigations already indicate that, here too, useful results may be expected.

The region now to be considered offers far greater difficulties than that treated in *Mount Wilson Contribution* No. 82. These are a consequence of the proper motions which, in general, are

[1] *Contributions from the Mount Wilson Solar Observatory*, No. 147.

[2] Research Associate of the Carnegie Institution of Washington, Mount Wilson Solar Observatory.

[3] A small supplementary group of 8 stars between $b=-30°$ to $-42°$, $l=156°$ to 178°, was added for a particular purpose (see Sec. 8).

exceptionally small and, superficially, show scarcely any recognizable regularity in their directions. Under the circumstances it may seem a bold undertaking to subject these stars to a treatment which in general is applicable only to motions approximately parallel and equal. One circumstance, however, proves that below the apparent irregularity there must be hidden a real similarity: the radial velocities of the stars throughout the region vary relatively little. Hence (*a*) the stars must lie near the antivertex[1] and (*b*) the range of real velocity must be small.

Circumstance (*a*) explains why the proper motions are so small and makes large differences in the position angles of the motions appear as a necessity; (*b*) makes it all but certain that the seeming irregularities in the proper motions are not greater than might be expected among stars moving along approximately parallel lines with approximately equal velocities. As a matter of fact, investigation shows that the proper motions, instead of contradicting the conclusions as to the approximate equality of motion derived from the radial velocities, prove that the spectroscopic velocities give a very inadequate idea as to the degree of this equality. The seeming irregularity of the motions is due to the fact that the systematic part (stream-motion) is to a great extent obliterated by foreshortening, while the irregular part (peculiar motion) shows as strongly as anywhere else.

As the present investigation furnishes the parallaxes of practically all the B stars within the region considered, a map showing the positions of the stars in space might have been given, as was done for longitudes 216°–360° in *Mount Wilson Contribution* No. 82. I have not done so, in the hope that further investigation will furnish data for extending this map to include all of the B stars between $-30°$ and $+30°$ of galactic latitude.

2. THE MATERIALS

In the main I have used (as in *Mount Wilson Contribution* No. 82) all the B stars in Boss's catalogue, 168 in all. The spectra

[1] The name "antivertex," in analogy with the name "antiapex," is used for the point around which the motions *converge*, whereas the name "vertex" is reserved for the divergent.

and the magnitudes have been taken from *Harvard Annals*, **50**, the proper motions from Boss. Besides these bright stars I had the good fortune to be able to include all the faint B and A stars within the limits

$$\text{R.A. } 5^{h}0^{m} \text{ to } 6^{h}0^{m} \quad \text{Dec. } -10^{\circ} \text{ to } +20^{\circ} \tag{1}$$

for which Miss Cannon has recently determined the spectra. This list contains 412 B and 760 A stars, many of them fainter than magnitude 9.0, some even fainter than 9.5.

It is a pleasure to me to extend my warmest thanks both to Professor Pickering and to Miss Cannon for their generous help and courtesy. In using their results I have had a foretaste of the vast possibilities that will be afforded by the completion of the undertaking known as the *Revised Draper Catalogue*.

3. THE NEBULA-GROUP

A glance at the lower map of Plate IX in *Mount Wilson Contribution* No. 82 shows that the bright B stars between $5^{h}20^{m}$ and $5^{h}40^{m}$, on both sides of the equator, are particularly crowded. This crowding, however, is shown with perfect distinctness only by the Pickering-Cannon stars. Defining the position of the group by hour- and parallel-circles we have the limits at

$$\text{R.A. } 5^{h}13^{m} \text{ and } 5^{h}40^{m} \quad \text{Dec. } -9^{\circ} \text{ and } +5^{\circ} \text{ (1900).} \tag{2}$$

With a more complicated boundary-line we might diminish the area by about 20 per cent.

Within this boundary the B0–B5 stars are about 12 times and the B8–B9 stars about 5.7 times more numerous than in the surrounding regions. This alone proves, incontestably in my opinion, that we have to do with a local group which probably does not extend in depth much farther than it does laterally.

We have still further proof. The bright B stars, the stars of our list in Boss's catalogue within the limits (2), are almost exclusively of spectral types B0–B3, whereas outside the group they include a considerable number of B5–B9 stars. Omitting one star marked B whose subclass is unknown, I find the following numbers of stars:

	Oe5–B3	B5	B8–B9	
Within limits (2)	23	1	1	(P)
Outside limits (2)	80	21	41	

Add to this the facts that, as will presently appear, the radial velocities of the Oe5–B3 stars are very nearly equal, and consequently that the component u[1] of the peculiar motion is exceptionally small; that the component τ of the proper motion, as already explained, is not greater than the observational errors—all of which proves that the real motions of these stars are very nearly parallel and equal, much more so than is the case with the stars outside the limits (2); and there can be no doubt that we have to deal with a local group, somewhat of the nature of the Pleiades or the Hyades. As this local group surrounds the famous Orion Nebula, which very probably, though we cannot yet say certainly, forms part of the group, I will call it the Nebula-group.

4. SYSTEMATIC ERROR IN BOSS'S μ_α AND μ_δ

I will begin by studying the stars outside the Nebula-group. But such a study must of necessity be preceded by an investigation of the systematic errors in the proper motions of Boss's catalogue. In the case of proper motions so exceptionally small (a total motion as great as $0\overset{''}{.}04$ is found for only 6 per cent of our stars), such errors are of very considerable importance. It will be our aim not only to find plausible values for them, but to arrange all our calculations in such a way that the residual uncertainty has a minimum effect.

The determination of these errors will be based on the hypothesis that the direction of motion of the stars of small proper motion is not systematically different from that of the stars whose motions are somewhat more considerable. If the systematic motion of the B stars is wholly a parallactic displacement, as is generally assumed, the hypothesis will be justified. The same will be true if all the stars considered have the same stream-motion; only in the case of different stream-motions will it fail. To diminish the probability of such a state of affairs, and to increase the number of objects on which we can base our conclusions, stars in neighboring parts of the sky and of different spectra were included in the investigation.

The method itself I have explained in several places, for example, *Mount Wilson Contribution* No. 82, p. 48. Let p_1 represent the

[1] For the meaning of our notation see *Mount Wilson Contr.* No. 82, p. 77.

average position angle of the motions of stars of small proper motion, p_2 the same quantity for the stars of large proper motion, and suppose the two reduced to the same point of the sky. If then p_1 and p_2 differ, the difference will be attributed to systematic errors in μ_α and μ_δ. The equation of condition given in *Mount Wilson Contribution* No. 82 supposes that the corrections $\delta\mu_\alpha$ and $\delta\mu_\delta$ are small as compared with the motions μ_α and μ_δ themselves. In the present case, where the proper motions are as a rule so exceedingly small, the validity of such a supposition seems doubtful, and it will be preferable to use the equation in its rigorous form:

$$\frac{\mu_2 \cos p_2 - \mu_1 \cos p_1}{\mu_1\mu_2}\delta\mu_\alpha - \frac{\mu_2 \sin p_2 - \mu_1 \sin p_1}{\mu_1\mu_2}\delta\mu_\delta = \sin(p_2 - p_1) \qquad (3)$$

I began with the four regions (A) including most of the stars contained in the present paper, with the exception of the Nebula-group and its nearest surroundings.

Group	1	l*	150°	to	165°	b −12°	to	+ 4°	
"	2	"	200	"	216	" −10	"	+ 3	(A)
"	3	α	3^h30^m	"	5^h12^m	δ −17	"	+ 3	
"	4	"	6 10	"	8 10	" −16	"	+10	

In order to increase the weight of the determination somewhat, I included not only the B stars, but also the A0–A3 stars with total proper motion below 0″.100. The spectra B0–B8 and B8–A3 were treated separately at first, but as no real difference seemed to exist the results were combined. The mean right ascensions and declinations of the stars of large and small proper motion were of course found to be slightly different. The reduction to the same point was made by adopting for the vertex the provisional position 17^h48^m, $+9°$. The results obtained are given in Table I.

Substituting in (3) we find

			Average δ	No. Stars	Adopted Weight	
$-48.1\,\delta\mu_\alpha$	$-\ 8.6\,\delta\mu_\delta$	$=+0.078$	+17°	38	9	
+29.9	+50.2	$=+0.485$	−27	31	6	(4)
−70.7	−14.9	$=-0.309$	− 3	17	2	
−39.1	+44.2	$=+0.156$	− 4	38	4	

* The inconsistency in giving the limits of part of the regions in galactic longitude and latitude and of the other part in right ascension and declination is due to the later introduction of the third and fourth regions, which at first were left out of consideration.

A solution by least squares leads to

$$\left.\begin{aligned}\delta\mu_\alpha &= +0''.0004 \pm 0''.0019 \\ \delta\mu_\delta &= +0.0072 \pm 0.0024\end{aligned}\right\} \qquad (5)$$

With respect to the probable errors (last column of Table I), this solution represents the observations satisfactorily. We thus

TABLE I

AVERAGE POSITION ANGLE OF μ

Group	α	δ	100μ	p	r	No. Stars	p_2-p_1 obs.	p_2-p_1 comp.	O−C	P.E.
1	5^h43^m	$+17°$	1''.20 2.89	162°.0 166.5	7°.5 2.0	11 27	+4°.5	− 3°.5	+ 8°.0	± 7°.7
2	7 21	−27	1.18 3.05	256.0 285.0	8.0 4.0	14 17	+29	+21.5	+ 7.5	9.0
3	4 43	− 3	0.98 3.28	182.0 164.0	20.0 13.0	7 10	−18	− 6.5	−11.5	23.8
4	6 53	− 4	1.19 3.92	215.5 224.5	15.0 6.0	15 23	+ 9	+18.5	− 9.5	±16.2
					Total	124				

find a vanishing correction for the proper motion in right ascension, but a considerable one for that in declination. Reasons will be found for assuming that the error is greatest between declinations $-15°$ and $-35°$. For these we have, exclusively by the second group, putting $\delta\mu_\alpha = 0''.000$,

$$\delta\mu_\delta = +0''.0097 \qquad (\text{Dec. } -27°) \qquad (6)$$

This result, if well established, would be so important that I have sought for further evidence. For reasons already given, there will be advantages in including stars of neighboring regions and of different spectra. The following represents what I have been able to find.

a) *Astronomische Nachrichten*, **160,** 338, 1903.—From Table IV of this article we find the results in Table II. The values $\delta\mu_\delta$ Kapt. were obtained on the hypothesis adopted above. There is a very serious difference in Zone $-18°$ to $-36°$ between the proper motions from Boss and those obtained by adopting our hypothesis. For

the present purpose we derive from Table II: Correction to Boss's μ_δ, between the limits $\alpha=0^h$ to 24^h, $\delta=-18°$ to $-36°$,

$$\delta\mu_\delta=\frac{+0''.42-(-1''.17)}{100}=+0''.016. \tag{7}$$

It is worth remarking that for Auwers' proper motions the correction is small; in fact, does not exceed the limits of its uncertainty.

TABLE II

Zone	100 $\delta\mu_\delta$		
	Kapt.–Cape	Boss–Cape	Auw.–Cape
0° to −18°..	....	+0″.11	....
−18 " −36 ..	+0″.42±0″.24	−1.17	+0″.12±0″.17 (133)
−36 " −54 ..	−0.22±0.19	−0.27	+0.30±0.19 (100)
−54 " −72 ..	+0.37±0.22	+0.83	+0.91±0.26 (57)

b) *Groningen Publications*, No. 21, p. 40 (34).—Weersma finds, by the principle here used, for Zone $\delta=-20°$ to $-40°$

$$\mu_\delta \text{ Kapt.}-\mu_\delta \text{ Newcomb}=+0''.0087 \tag{8}$$

where μ_δ Kapt. again stands for the proper motion in δ corrected by the foregoing hypothesis. By a direct comparison of Newcomb's and Boss's catalogues between $\alpha=4^h.5$ to $8^h.5$, $\delta=-5°$ to $-35°$ I find

$$\mu_\delta \text{ Newcomb}-\mu_\delta \text{ Boss}=+0''.0033 \text{ (34 stars)}. \tag{9}$$

From (8) and (9)

$$\delta\mu_\delta=\mu_\delta \text{ Kapt.}-\mu_\delta \text{ Boss}=+0''.0120. \tag{10}$$

c) *G, K, M stars with proper motions* $\geqq 0''.017$.—The correction (5) depends on a comparison of stars of proper motions $\leqq 0''.016$ with those having greater motion. If the correction is confirmed by stars with proper motions $\geqq 0''.017$, the probability that we have to deal, not with systematic proper motion, but with systematic catalogue error, will be enormously increased, especially if we use stars of quite different spectra. It is desirable that the investigation cover the regions (A); but it is important, first, to increase the number of stars by extending these limits, and, second, to avoid the neighborhood of the antiapex (6^h, $-34°$). I therefore

selected the region R.A. 7^h to 12^h, Dec. $-15°$ to $-40°$ and considered two classes of stars,

$$a)\ 0''.017 \leqq \mu \leqq 0''.050, \qquad b)\ \mu \geqq 0''.070.$$

All the Boss stars of spectrum G, K, M, and, further, those of unknown spectrum were used. I was thus led to the enormous difference:

$$\delta\mu_\delta = +0''.026 \quad \text{(69 stars)} \tag{11}$$

No great weight can be attributed to this result, because the values of p vary so much that in a few cases there is uncertainty whether they ought not to be increased by 360°.

d) ***Finally by a direct comparison*** of Boss's catalogue with the fundamental catalogues of Auwers and Newcomb for the region R.A. 4^h30^m to 8^h30^m, Dec. $-5°$ to $-35°$ I find

$$\left.\begin{aligned}\delta\mu_\delta &= +0''.0015 \quad \text{(33 stars) by Auwers}\\ &= +\ \ .003 \quad \text{(34 stars) by Newcomb}\end{aligned}\right\} \tag{12}$$
$$\delta\mu_\delta = +0''.002$$

Summarizing, we have Table III.

TABLE III

	Equation	Limits		$\delta\mu_\delta$
B–A3 (second group)	(6)	$7^h.0$ to $8^h.0$	$-24°$ to $-36°$	$+0''.010$
All spectra	(7)	0.0 " 24.0	-18 " -36	+ .016
All spectra	(10)	0.0 " 24.0	-20 " -40	+ .012
G, K, M, and unknown	(11)	8.0 " 12.0	-17 " -40	(+ .026)
Auwers, Newcomb	(12)	4.5 " 8.5	-5 " -35	+ .002
Boss	....	....	....	0.000

For stars in the positive declinations of the region under consideration the position angles p are little affected by a correction of μ_δ. Moreover, there is every reason to believe that the correction is small. For the declinations 0° to $-20°$, the last two equations of (4), putting $\delta\mu_\alpha = 0.000$, give $\delta\mu_\delta = +0''.0050$. Considering everything, I finally resolved to adopt the following values:

$$\left.\begin{array}{cc} \delta & \delta\mu_\delta \\ -20° \text{ to } -40° & +0''.008 \\ 0 \;\text{"}\; -20° & +0.004 \\ 0 \;\text{"}\; +20° & 0.000 \end{array}\right\} \tag{13}$$

Boss's proper motions in declination, thus corrected, are given in Table XXXIX and have been used in all the discussions. His proper motions in right ascension were adopted uncorrected.

5. FIRST DETERMINATION OF THE ELEMENTS

For all the B stars in each of the regions (A) averages were found for α, δ, μ_α, μ_δ, with the results shown in Table IV. A second summary including only stars with proper motions $\geqq 0''.017$ is given in Table V.

TABLE IV

AVERAGES FOR ALL B STARS

Group	α	δ	No.	μ_α (Great Circle)	μ_δ	v Vert. (14)	p obs.	p comp. Vert. (14)	p comp. Vert. (24)	λ Vert. (24)
1	5^h46^m	$+17°.0$	23	$+0''.0031$	$-0''.0186$	$0''.0189$	$170°.5$	$172°.5$	$180°$	$152°.0$
2	7 18	−27.6	25	−0.0129	+0.0094	0.0160	306.0	310.5	305	152.4
3	4 20	− 4.7	12	+0.0081	−0.0017	0.0083	102.0	103.0	108	158.3
4	6 47	− 5.8	18	−0.0132	−0.0063	0.0146	244.0	251.0	253	163.5

TABLE V

AVERAGES FOR B STARS HAVING $\mu \geqq 0''.017$

Group	α	δ	No.	p obs.	p comp. Vert. (14)	p comp. Vert. (24)
1	5^h52^m	$+17°.2$	15	$172°.6$	$176°.0$	$184°$
2	7 19	−27.3	15	303.4	309.0	302
3	4 17	− 3.1	9	104.3	106.5	111
4	6 49	− 3.7	9	249.0	242.0	245

If the stars in these regions have the same motion in space, the directions defined by the values of p must intersect in a single point. This is indeed the case with extreme approximation, the position of the antivertex being:

$$\left.\begin{array}{ll}\text{From Table IV} & 6^h\ 0^m,\ -10°.5 \\ \text{From Table V} & 5\ 57\ ,\ -10\end{array}\right\}\ \text{Adopted } 6^h 0^m,\ -10° \qquad (14)$$

All the directions of Table IV pass this point within a distance of about 2°; those of Table V within 2°.5.

This close convergence to a single point makes the community of motion in the four regions highly probable. Assuming this coincidence of motion for the moment, I determined the stream-velocity from the radial velocities. On account of the scarcity of data I extended the regions a little, using the limits:

$$\left.\begin{array}{lllllll} \text{Group } 1 & \text{R.A.} & 5^h12^m & \text{to } 6^h30^m, & \text{Dec.} & +10^\circ & \text{to } +30^\circ \\ \quad " \;\; 2 & " & 5\;45 & "\;\; 7\;45 & " & -36 & "\;\; -20 \\ \quad " \;\; 3 & " & 4\;\;0 & "\;\; 5\;12 & " & -16 & "\;\; +15 \\ \quad " \;\; 4 & " & 5\;43 & "\;\; 8\;10 & " & -19 & "\;\; +10 \end{array}\right\} \qquad (15)$$

For the average radial velocities, first applying the constant correction -4.3 km (see *Mount Wilson Contribution* No. 82, p. 28), we find the results in Table VI.

TABLE VI

RADIAL VELOCITIES

Group	α	δ	λ Vert. (24)	No.	$\rho-4.3$	ρ comp. Vert. (24)	O−C	P.E.
					km	km	km	km
1........	5^h44^m	$+18°8$	150°	9	+16.1	+17.3	−1.2	±1.8
2........	6 39	−29.4	157	9	+23.3	+18.4	+4.9	±1.5
3........	4 39	− 1.3	161	9	+14.7	+18.9	−4.2	±0.8
4........	6 42	−10.5	166	11	+21.9	+19.4	+2.5	±1.9

From this we compute the stream-velocity V by

$$V \cos \lambda = \rho , \qquad (16)$$

in which λ represents the angular distance from the antivertex

$$5^h44^m, \qquad -11^\circ \qquad (17)$$

which differs only slightly from (14) and will later be accepted as the definitive position. A least-squares solution leads to

$$V = 20.55 \text{ km} \quad (38 \text{ stars}). \qquad (18)$$

Outside the limits (15) there are three stars in our list with known radial velocities. Including these,

$$V = 20.0 \pm 1.5 \text{ km} \quad (41 \text{ stars}). \qquad (19)$$

Finally we compute the average parallax by (see *Mount Wilson Contribution* No. 82, p. 35)

$$\bar{\pi}=\frac{\bar{v}}{0.212\ V \sin\lambda}. \tag{20}$$

Adopting (17) and (19) we obtain from the data of Table IV

$$\bar{\pi}=0''.0083\pm0''.0008 \quad (78 \text{ stars}). \tag{21}$$

6. COMMUNITY OF MOTION OF THE FOUR GROUPS

It was remarked that the intersection of the four lines defined by the values of p in Tables IV and V is a strong argument in favor of the supposition that the four groups have the same motion. There are, however, other criteria. If there is community of motion, and if we may assume that the values of the mean parallax are about the same for the four groups, the values of v in Table IV must be proportional to sin λ. With vertex (14) the proportionality, as indicated by Table VII, is not close. It can be greatly

TABLE VII

Group	v obs.	0''.0351 sin λ Vert. (14)	O−C		
			Vert. (14)	Vert. (22)	Vert. (24)
1	0''.0189	0''.0164	+0''.0025	+0''.0027	+0''.0026
2	0.0160	0.0148	+0.0012	−0.0026	−0.0014
3	0.0083	0.0154	−0.0071	−0.0006	−0.0046
4	0.0146	0.0079	+0.0067	+0.0005	+0.0035

improved by diminishing the right ascension of the antivertex, and would be quite satisfactory (see Table VII) if we adopted

$$5^h12^m, \qquad -10^\circ \tag{22}$$

instead of (14). This, however, would not satisfy the directions of Tables IV and V, and we must accept a compromise between the two positions.

Adopting for each of the four regions the values

$$V=20.0 \text{ km}, \quad \bar{\pi}=0''.0081, \tag{23}$$

the latter of which slightly diverges from (21) but agrees with the definitive value adopted later, we can compute for each of the four

regions a separate vertex. Table IV furnishes for the centers of each of the regions the direction of the antivertex, whereas its distance λ can be found from v by (20) and (23). I thus find for the antivertices:

Group	α	δ	No.	Deviation
1..........	$6^h\ 6^m$	$-15^\circ.8$	23	$8^\circ.0$
2..........	5 50	− 9.3	25	4.3
3..........	5 18	− 7.3	12	6.9
4..........	5 12	−16.0	18	7.3
Mean...	5 37	−12.2	....	....

The last column shows the distance of the individual antivertices from the mean of all. Taking into account the small number of stars from which these antivertices were obtained, and considering that the supposed absolute equality of $\bar{\pi}$ for the four groups is highly improbable, the deviations are not greater than might reasonably have been expected. Certainly they afford no sufficient reason for doubting the community of motion.

As the position (14), which rests on the directions in Table IV and is thus independent of any supposition as to the equality of the values of $\bar{\pi}$, whereas this supposition is involved in the result obtained just now, I will adopt as *final* the intermediate position of the antivertex

$$5^h 44^m, \qquad -11^\circ \tag{24}$$

The values of p computed with the aid of (24) have been inserted in Tables IV and V. To my regret the computed values of p and λ given in Table XXXIX have been derived with a slightly different antivertex, viz.,

$$5^h 48^m, \qquad -9^\circ. \tag{25}$$

I have not judged it worth while to repeat the computations because the two positions probably agree within the uncertainty of (24).[1]

[1] Further, (24) is probably also to be considered as only provisional. A preliminary investigation seems to show that the community of motion extends much beyond the limits adopted in the present paper. If this proves to be true, it will be necessary later to treat all the stars with common motion as a whole, which will probably change the definitive elements a little.

7. COMMUNITY OF MOTION, CONTINUED

A more serious objection to the assumption of community of motion is furnished by the radial velocities. If from the data in Table VI we compute the stream-velocity V separately for the four regions we obtain

Group 1	$l=157°$,	$V=-18.6\pm2.1$ km	No. = 9	(26)
" 2	205	-25.3 ± 1.6	9	
" 3	164	-15.6 ± 0.85	9	
" 4	186	-22.6 ± 2.0	11	

Comparing the results of the second and third groups we find a difference of

$$9.7\pm1.8 \text{ km}, \tag{27}$$

which seems real enough. At all events it becomes necessary to examine the matter more closely.

I first made a brief preliminary investigation for the region adjoining those of the present paper on the side of the small galactic longitudes (120° to 150°). It turned out that the direction of motion here passes very near the point (24), which satisfies the directions of all four regions under consideration. For the radial velocity there is a decidedly closer approach to the first and third regions than to the other two, which lie on the side of greater longitudes—see (26)—the part of the sky investigated in *Mount Wilson Contribution* No. 82. It would therefore excite no particular surprise if we found the velocity, or even all the elements of motion, to diverge somewhat from those of the other regions toward those of the region treated in *Mount Wilson Contribution* No. 82. As a matter of fact, we really find the reverse. Indeed, the observations in the second and fourth regions, badly as they are represented in radial velocity by the elements (19) and (24), are still much better represented by these elements than by those for the higher galactic longitudes, namely,

$$\text{Vertex, } 18^h24^m, \quad +39°; \quad V=-18.0. \tag{28}$$

The results of the comparison are

Group	p obs.	ρ obs. -4.3	Computation By				O−C			
			(19) and (24)		(28)		(19) and (24)		(28)	
			p	ρ	p	ρ	p	ρ	p	ρ
2.........	306°	+23.3	305°	+18.4	221°	+17.7	+1°	+4.9	+85°	+5.6
4.........	244	+21.9	253	+19.4	188	+15.7	−9	+2.5	+56	+6.2

The position angles are therefore not at all represented by (28), and the radial velocities markedly worse than by (19). The case therefore stands thus: If we accept the reality of the divergence of the radial velocity of Group 2, this group can be considered as local only, for it stands apart from the stars, both in lower and in higher galactic longitudes. If we do this, there is every reason for also including the fourth region in the local group, which thus becomes extensive enough for a determination of all its elements.

The directions furnished by Table IV for Groups 2 and 4 cut each other at an angle of 65°; those for Groups 1 and 3, at an angle slightly greater. If therefore we take the points of intersection as the antivertices, they must in both cases be considered well determined. Both agree almost perfectly with position (14) obtained from the four regions together. Accepting these vertices, the stream-velocity is obtained by the data of Table VI. I find

$$\left.\begin{array}{lll} \text{Groups 2 and 4} & V=23.5\pm 1.2 & \text{(20 stars)} \\ \text{“ \quad 1 “ \quad 3} & \underline{V=17.2\pm 1.2} & \text{(18 stars)} \\ \text{Difference} & 6.3\pm 1.7 & \end{array}\right\} \qquad (29)$$

The result thus would be that the second and fourth regions form a local group whose direction of motion coincides absolutely with that of the others but whose velocity is 37 per cent greater.

Some further criteria may be considered:

1. A local group can usually be seen as such on an ordinary star map. If on the map in *Mount Wilson Contribution* No. 82

($\mu \leqq 0''.016$) we exclude the Nebula-group (2) there seems to be indicated a group within the limits, approximately,

$$l = 180^\circ \text{ to } 216^\circ; \quad b = -30^\circ \text{ to } +4^\circ, \tag{30}$$

which covers Group 2 and the richer part of Group 4.

2. In a local group we expect the proper motions to be nearly equal. The proximity of the antivertex and the minuteness of the proper motions, which result in their being strongly influenced by observational errors, and finally the fact that if the group is not very condensed many extraneous stars will be seen projected on it, make the criterion less effective in the present case than ordinarily. If we call proper motions $\leqq 0''.016$ small, then, within the limits (30), 61 per cent of the motions are small; outside these limits, only 39 per cent.

3. In local groups we expect to find a distribution of spectra differing from that in the rest of the sky. Thus, in the Nebula-group we found a strong predominance of the Oe5–B3 stars (see Section 3). Within the limits (30) we find that 77 per cent of all the stars are Oe5–B5; outside these limits there are 59 per cent.

4. For the mean parallax of the stars within the limits (30) we find, as above, $\bar{\pi} = 0''.0076$, which scarcely differs from (21) and (23).

5. For the Nebula-group we shall find further on $\bar{\pi} = 0''.0054$. Its stream-velocity can scarcely differ from the value in (19). If at all different, it seems to be rather smaller. The elements for Groups 2 and 4 do not approach those of the Nebula-group more than they do those of Groups 1 and 3.

In conclusion, we cannot deny the possibility of a local group; still, in my opinion, the similarity in the direction of the motions outweighs all other arguments, so that the probability is not in favor of such a group. The difference (29) of 6.3 km between the stream-velocities of Groups 2+4 and 1+3, which is the main argument in favor, is indeed almost four times the probable error. But such a result is perhaps not so surprising for radial velocities as it would be for other quantities, because there must still be numerous cases of undiscovered orbital motion.

In what follows we shall assume that the stream-motions of all four groups are the same. The question can probably be settled

without difficulty by the fainter stars as soon as the *Revised Draper Catalogue* is published. Even if it should then be proved that there really is a local group, our results will be little altered. The parallaxes, and consequently the absolute magnitudes, M, in the main will not be altered, and the same, of course, holds for the luminosity curve. The computed values of p also will not change. Only the values of the radial velocities will be somewhat different, for instead of computing these with (19) we shall have to use (29).

The average values of the parallax finally adopted are in some cases slightly different from the mean in (23), because, first, our four groups do not include all the available material, and, secondly, it does not seem reasonable to adopt exactly the same value for all the partial groups. From the new computation Boss 1401, 1517, and 1994 were excluded—the first and the last because they do not seem to belong to the same system as the rest of the stars (as proved by the values of p), and No. 1517 because it has so exceptionally large a proper motion. The values thus obtained for the groups of different λ are given in Table VIII. For certain purposes they have been further contracted into the considerably overlapping groups in Table IX.

TABLE VIII

λ	Mean λ	$\bar{\pi}$	No. of Stars	Weight
145° to 149°...	147°	0″.0081	26	138
150 " 154 ...	152	0.0075	40	160
155 " 159 ...	157	0.0107	28	77
160 " 164 ...	162	0.0066	17	29

TABLE IX

λ	Mean λ	$\bar{\pi}$	No. of Stars	Weight
145° to 154°...	150°	0″.0077	66	11
150 " 159 ...	154	0.0087	68	9
152 " 167 ...	157.5	0.0083	75½	4

From Table VIII, I find, in good agreement with (23), for the general average:

$$\bar{\pi} = +0''.0081 \pm 0''.0007 \quad \text{(111 stars).} \tag{31}$$

8. INFLUENCE OF SYSTEMATIC ERROR AND SUMMARY OF DEFINITIVE ELEMENTS

On account of the excessive smallness of the proper motions and the consequently strong influence of observational errors, it will not be deemed superfluous if we try to find the effect of any remaining traces of systematic error. For this purpose I made new solutions with the data in Tables IV and VI on the supposition: (*a*) that Boss's values of μ_α require a constant correction $\delta\mu_\alpha = +0''.0030$; (*b*) that the values of μ_δ in Table XXXIX, that is, of Boss's μ_δ corrected by (13), require a further correction $\delta\mu_\delta = +0''.0030$; (*c*) that the correction (13) for μ_δ is wholly false, our new solution thus, in this case, starting with the uncorrected μ_α and μ_δ of Boss.

For this special investigation I have disregarded the values of the v in the derivation of the vertex and adopted as antivertex the point toward which all the directions converge with the greatest approximation. I thus found

Case		Antivertex		$\bar{\pi}$	V	
(1)	No correction	$6^h\ 0^m$	$-10^\circ.5$	$0''.0082$	20.8	(32)
(2)	Correction (*a*)	6 12	− 8	0.0080	21.3	
(3)	" (*b*)	5 55	− 4	0.0082	20.8	
(4)	" (*c*)	5 52	−18	0.0076	20.55	

In every case the four directions of Table IV intersect nearly at a single point. This is most perfectly realized in the cases (1) and (2), where none of the directions deviate from the adopted antivertex more than 2°. In cases (3) and (4) the greatest distance is 4° or 4°.5.

In view of these results I think we may conclude that, as a consequence of the remaining systematic errors and the consideration of the values of v in Section 6, it is not impossible that there may still be an error of 3° or 4° in the position of the antivertex, but that the resulting uncertainties in V and $\bar{\pi}$ must be negligible.

This freedom of V from systematic error is due to the fact that the region as a whole is so near the antivertex, and to the further fact that the four separate regions lie fairly symmetrically around the antivertex. As for π, its freedom from systematic error is wholly attributable to this last circumstance. It was this consideration which in great part led to our choice of the four regions

and compelled us to introduce a supplementary group outside the limit at galactic latitude $-30°$.

The danger from the systematic errors being thus in great measure avoided, there is every reason to accept the probable errors as a measure of the accuracy obtained. Collecting results, we have as *definitive elements:*

$$\left.\begin{array}{l}\text{Vertex } 17^h44^m, \quad +11° \quad (\text{uncertainty } \pm 3° \text{ or } \pm 4°) \\ V=-20.0\pm 1.5 \text{ km (P.E.)} \\ \bar{\pi}=0\overset{''}{.}0081\pm 0\overset{''}{.}0007 \text{ (P.E.)}\end{array}\right\} \quad (33)$$

This will not prevent the adoption for π of the separate values of Table IX. It is a very significant fact, to which I hope to revert in a later publication, that this vertex coincides so nearly with that of the first stream of all the non-helium stars, for which Eddington in *Monthly Notices*, **71**, 42, 1910, finds 18^h3^m, $+14\overset{\circ}{.}6$.

9. OBSERVED DISTRIBUTION OF THE VALUES OF v

Now that a good estimate of the average π has been obtained, we have to find the range in distance. It would be most desirable to determine the individual parallax of each star, but as this is not feasible, we will try to derive mean parallaxes for stars having given values of λ and v and, further, the probable deviation of the individual parallaxes from this mean. The amount of this deviation as a fraction of the parallax will determine the confidence with which we may use the mean parallax as a substitute for the individual parallaxes.

I will begin with the consideration of certain necessary data: first of all, with an examination of the distribution of the values of v for different values of λ. By countings in Table XXXIX, omitting stars within 15° of the antivertex and excluding Boss 1517, I find the data in Table X.

The number of stars being so small, pains have been taken to smooth thoroughly the results. We first contract the data into three partially overlapping zones:

λ	$\bar{\lambda}$	$\bar{v}$	No.	
145° to 154°	150	0″.0164	66	(34)
150 " 159	154	0.0161	68	
152 " 167	157.5	0.0135	75½	

Even so, it is not easy to arrive at an entirely satisfactory result. The difficulty would be greatly reduced if we knew the form of the frequency curve for the v. If there were no observational errors,

TABLE X

DISTRIBUTION OF THE VALUES OF v

100 v	λ						Totals
	145°–149°	150°–151°	152°–154°	155°–159°	160°–164°	165°–167°	
6″.0 to 7″.0..	....	1	....	....	....	....	1
5.0 " 6.0..	....	....	....	....	....	....	....
4.0 " 5.0..	1	....	1	....	....	1	3
3.0 " 4.0..	5½	2	1	7	1	....	16½
2.0 " 3.0..	6½	5	4	8½	1	....	25
1.0 " 2.0..	6	3½	9	1½	4	2	26
0.0 " 1.0..	5	2½	7	9	7	½	31
0.0 " −1.0..	2	1½	1½	1	4	2½	12½
−1.0 " −2.0..	....	....	....	½	....	....	½
<−2.0....	....	....	1	½	....	....	1½
Totals......	26	15½	24½	28	17	6	117

such a form could be assigned, at least if we assumed—on grounds presently to be given—that the values of u (component of *linear* peculiar motion) are distributed according to an error-curve. For it follows from formula (20) that, for any definite value of π,

$$\pi = 0.212\, v \cdot V \sin \lambda \tag{35}$$

with individual deviations following an error-curve having the probable error

$$r_v = \pm 0.212\, \pi r_u ,$$

where r_u is a constant for which later will be found the value 1.67 km. The deviations of v are thus seen to be proportional to v. From the theory developed elsewhere[1] it follows that the values of v, for the same value of λ, will be distributed according to the formula

$$\text{Prob.}\,{}_{v}^{v+\delta v} = \frac{h}{\sqrt{\pi}} \cdot \frac{\text{Mod.}}{v} e^{-h^2[\log v - M]^2} \delta v , \tag{a}$$

that is, the values log v will be distributed according to an error-curve.

[1] *Skew Frequency Curves in Biology and Statistics*, Noordhoff (Groningen), 1903, p. 21.

This, however, holds only in the absence of observational errors. Since these follow an error-curve, the distribution of the observed values of v will be something between (a) and an ordinary error-curve. In particular the real curve, unlike (a), will yield certain frequencies for negative values of v.

Now we obtain an intermediate curve if in (a) we substitute $v+K$ (K being a constant) for v. The formula then becomes

$$\text{Prob.}\,{}_{v}^{v+\delta v}=\frac{h}{\sqrt{\pi}}\cdot\frac{\text{Mod.}}{v+K}e^{-h^2[\log(v+K)-M]^2}\delta v, \tag{b}$$

which implies that the distribution of $\log(v+K)$ follows an error-curve.

Led by these considerations I have tried to learn whether it is possible to determine the three constants in (b) in such a way that the observed frequencies are well represented. This being really the case, I have considered the best representation by (b) as that to be adopted. The derivation of the best-fitting values of h, K, M is very easy.[1] It is to be recommended that this be made in such a way that the arithmetical mean values $\bar{v}$ in (34) are perfectly represented. This gives as a first condition between the constants

$$\log(\bar{v}+K)=M+\frac{1}{4h^2\cdot\text{Mod.}}\,. \tag{36}$$

The values obtained are as follows:

λ	150°	154°	157°5	
h	9.347	15.274	6.280	
K	0.0608	0.1095	0.0451	(37)
M	−1.1190	−0.9033	−1.2464	

These being substituted in (b), we find by integration (which offers no difficulty) the values in the second, fifth, and eighth columns of Table XI. Those in the other columns will be explained in Section 13.

10. OBSERVED DISTRIBUTION OF THE VALUES OF τ

In order to increase the reliability of the conclusions, I have confined myself to the stars for which the probable error of τ does not exceed 0″0069. The number of stars is thus somewhat diminished,

[1] *Skew Curves*, art. 18, p. 34.

but this is not so material here, because stars at arbitrarily different distances from the vertex can be combined. Boss in his catalogue gives the probable error of 100 μ_α and 100 μ_δ. The mean of the two was combined to represent 100 times the probable error of the

TABLE XI

DISTRIBUTION OF THE VALUES OF v

(Theoretical Values are for $r_u = \pm 2.5$)

λ	$\bar{\lambda}=150°$			$\bar{\lambda}=154°$			$\bar{\lambda}=157°.5$		
	Obs.	Theor.	O−C	Obs.	Theor.	O−C	Obs.	Theor.	O−C
+0".055	0.5	0.4	+0.1	0.3	0.3	0.0	1.1	0.9	+0.2
.050	0.5	0.5	0.0	0.5	0.4	+0.1	0.7	0.7	0.0
.045	0.9	0.9	0.0	0.8	0.7	+0.1	1.1	1.0	+0.1
.040	1.6	1.7	−0.1	1.6	1.4	+0.2	1.6	1.6	0.0
.035	2.6	2.8	−0.2	2.7	2.3	+0.4	2.5	2.4	+0.1
.030	4.0	4.3	−0.3	4.4	3.9	+0.5	3.6	3.4	+0.2
.025	6.1	6.2	−0.1	6.5	5.9	+0.6	5.1	4.8	+0.3
.020	7.9	8.3	−0.4	8.4	8.1	+0.3	6.8	6.6	+0.2
.015	9.2	9.8	−0.6	9.9	10.0	−0.1	8.6	8.8	−0.2
.010	10.2	10.0	+0.2	10.2	10.8	−0.6	10.0	10.7	−0.7
+0.005	9.2	8.8	+0.4	9.0	9.8	−0.8	10.5	11.5	−1.0
0.000	6.7	6.3	+0.4	6.7	7.4	−0.7	9.7	10.2	−0.5
−0.005	4.1	3.6	+0.5	4.2	4.4	−0.2	7.3	7.1	+0.2
−0.010	1.8	1.6	+0.2	2.1	2.0	+0.1	4.5	3.8	+0.7
−0.015	0.6	0.7	0.0	0.8	1.0	+0.1	1.9	1.5	+0.4
−0.020	0.1			0.3			0.5	0.6	−0.1
Total	66.0	65.9	....	68.4	68.4	....	75.5	75.6	......

proper motion in any co-ordinate, hence also of 100 τ. This is given in Table XXXIX under the heading 100 r. I have further disregarded the stars within 20° of the adopted antivertex (24), which excludes the Nebula-group. The results of the countings are given in Table XII.

TABLE XII

NUMBER OF VALUES OF τ

τ	Observed Number	τ	Observed Number
<−0".025	0.5	+0.005 to + .010........	9.0
−0".020 to −0".025........	0.5	+ .010 " + .015........	8.5
− .015 " − .020........	2.0	+ .015 " + .020........	1.0
− .010 " − .015........	8.5	+0.020 " +0.025........	0.0
−0.005 " − .010........	13.5	>+0".025	1.0
.000 " − .005........	12.0		——
.000 " + .005........	15.5	Total................	72.0

The numbers fit very well an error-curve with the probable error $\pm 0''.0061$.

11. PROBABLE AMOUNT OF r_u AND DISTRIBUTION OF THE COMPONENTS OF THE PECULIAR MOTION

Let u and t[1] represent the components of the linear peculiar motion at right angles to the line of sight, one toward the antivertex, the other at right angles thereto; u and t subtend the angles υ and τ; $\bar{u}$ and $\bar{t}$ will denote the average values, all taken positively.

Since we suppose the directions of the peculiar motions to be distributed at random,

$$\bar{u}=\bar{t}=\text{average peculiar radial motion.} \tag{38}$$

This peculiar radial motion is the radial motion freed from stream-motion. We thus have to find[2]

$$\text{Observed radial motion}=-V\cos\lambda, \tag{39}$$

V and λ being in accordance with (33). The average of these values, all taken positively, will be $\bar{u}=\bar{t}$. Outside the Nebula-region there are 45 stars for which we have radial velocities. Of these, I exclude Boss 1761, 1817, 1935. The velocity of the first rests on measures of the bright H line, a determination, there is reason to think, not quite comparable with the others. The second is a spectroscopic binary; the velocity of the center of mass is a simple estimate which may prove to be largely in error. The velocity of the third is quite abnormal. Unless a spectroscopic binary, the star cannot belong to the same system as the other stars. For the remaining 42 stars

$$\bar{u}=5.78 \text{ km.} \tag{40}$$

Treating the stars of region (30) as a separate local group, this becomes 5.36. In *Mount Wilson Contribution* No. 82, p. 28, the corresponding value was found to be

$$\bar{u}=3.5 \text{ km} \quad \text{(61 full-weight stars)} \tag{41}$$

[1] t is the quantity for which in *Mount Wilson Contribution* No. 82 was used the somewhat less obvious notation v.

[2] The correction -4.3 km is first to be applied to the observed radial velocities.

Since the values of $\bar{u}$ depend partly on real peculiar motions and partly on observational errors, either one or the other or both of these factors must be greater now than in the earlier investigation.

The following considerations show that the difference is due to both causes. First, all results were included in (40), whereas in (41) the less reliable values were either excluded or admitted with diminished weight; secondly, (41) rests exclusively on Lick Observatory results, which do not include stars whose spectra are not susceptible of at least fairly good measurement, whereas the Mount Wilson observations, upon which (40) is partly based, include all objects on the original program, irrespective of their difficulty. If, to secure homogeneity, we limit ourselves to the values obtained by the Lick observers, and if further we exclude all values marked as uncertain, or given without any decimal, or as the estimated velocity of a spectroscopic binary, we find

$$\bar{u}=4.24\text{ km} \qquad r_u=\pm 3.6\text{ km} \qquad (16\text{ stars}) \qquad (42)$$

We thus approach the value (41), but the number of stars has become so small that little reliance can be put on the result.

Fortunately we can derive a much more reliable value in another way; but this requires a knowledge of the parallaxes, so that we shall have to anticipate to some extent results obtained later. With known parallaxes we can transform the τ components into linear motions by

$$t=\frac{\tau}{0.212\,\pi} \qquad (t\text{ in km per second}) \qquad (43)$$

In order to exclude completely the Nebula-group I use only stars for which $\lambda < 160°$. Since for objects of small parallax observational errors in τ appear much magnified in t, I include only stars for which $\pi \geqq 0\overset{''}{.}0070$. Finally, I exclude Boss 1517, for which the parallax is exceptionally large and uncertain.[1] For comparison the same computation was made for the stars in *Mount Wilson Contribution* No. 82. In this I avoided practically all excep-

[1] The value of τ, consequently of t, is quite normal for this star. Stars in our tables considered as not belonging to the system were, of course, also omitted; only one object, Boss 1944, was excluded on this account.

tional cases by limiting myself to the stars for which $\lambda \leqq 120°$ and galactic longitude $<320°$, and I tried to improve the results by omitting badly observed stars for which $100\, r > 0''.80$.

Before deriving averages I tried to find the frequency-curve of t freed from observational error. For the observed values[1] the distribution is at once given by counts, but we cannot hope to pass with any precision to the distribution of the true values as long as the observational errors predominate in the observed values of τ. This is the case if we treat all the stars, but it is no longer so if we confine ourselves to the best observed stars, those for which $100\, r \leqq 0''.40$. By this limitation the number of available objects becomes small, so that a very reliable result is not to be expected, but as a roughly approximate determination is to be preferred to none at all, or to a mere supposition, I do not hesitate to communicate my results.

We may safely assume, I think, that the frequency-curves, both of the observed and of the true values of t, are symmetrical. On this assumption we find, by counting, for the stars having $100\, r \leqq 0''.40$, the results in Table XIII. For both the Orion and the Scorpius-Centaurus region the observed distributions differ little from the normal error-law, as appears if we compare them with curves having probable errors of 2.32 and 1.83 km, respectively (third and sixth columns of Table XIII). The residuals show little that is systematic; they are mostly of different sign for the two regions. We thus conclude that the observed values of t are distributed closely in accordance with the error-law.

For the stars of other types, in particular for those of type K, there appears to be a deviation from this law in the direction of an excess of large motions.[2] We do not find much evidence of such an excess here, especially if we consider that the one star in the Scorpius-Centaurus region with a large value of t, namely, Boss 3115, has a very abnormal radial velocity. Should this be confirmed by later observations, the star should be excluded from the group.

[1] By *observed* values of t, I mean the values computed by (43) with the aid of the *observed* values of τ.

[2] *Proc. Nat. Acad. Sci.*, 1, 17–18, 1915.

Admitting, therefore, a distribution of the observed values of t according to the law of error, we should infer at once a similar distribution of the true values, could we assume that the observational errors affecting t also followed that law. Now this is

TABLE XIII

DISTRIBUTION OF $t_{obs.}$ ($100\ r \leqq 0''.40$)

$t_{obs.}$	Orion			Scorpius-Centaurus		
	No.	Normal $r=\pm 2.32$	O−C	No.	Normal $r=\pm 1.83$	O−C
0 to ±1km......	9	6.4	+2.6	6	7.2	−1.2
±1 " ±2	6	6.0	0.0	7.5	6.3	+1.2
±2 " ±3	4	5.0	−1.0	6	4.8	+1.2
±3 " ±4	1	3.8	−2.8	3.5	3.2	+0.3
±4 " ±5	2	2.7	−0.7	1	1.9	−0.9
±5 " ±6	3	1.9	+1.1	....	0.9	−0.6
±6 " ±7	1	1.0	0.0	....	0.4	
±7 " ±8	0.5	0.6	−0.1	....	0.2	
±8	1.5	0.6	+0.9	1	0.1	
Totals........	28	28.0	....	25	25.0	....

certainly not rigorously the case. We may assume that the observational errors of τ—consequently, by (43), those of t—follow the error-law *for stars of the same* π. For different values of π, however, the probable value of the observational error in t will be quite different; in fact, inversely proportional to π. Nevertheless, I convinced myself that the observational errors in t follow the error-law with sufficient approximation for our present purpose, and that the corresponding probable error is ± 1.47 km for the Orion region and ± 1.50 km for the Scorpius-Centaurus stars. Consequently, the distribution of the true values of t must also follow approximately the error-law, and the probable value r_t which equals r_u will be

$$\left.\begin{aligned} r_t=r_u&=\pm\sqrt{2.32^2-1.47^2}=\pm 1.80 \text{ (28 stars) Orion region} \\ r_t=r_u&=\pm\sqrt{1.83^2-1.45^2}=\pm 1.12 \text{ (25 stars) Scorpius-Centaurus} \end{aligned}\right\} \quad (44)$$

The smallness of these values is the most surprising and promising fact brought to light by the present investigation. The B stars of *Mount Wilson Contribution* No. 82 with those of the present

paper comprise about 65 per cent of all the B stars. With a value of r_u as small as those just found, the determination of accurate parallaxes of all these stars becomes merely a question of securing more accurate proper motions, which is only a matter of time; and with the powerful aid of photography the interval need not be so very long.

Further, preliminary investigation has shown that a similar parallelism exists for the B stars in the Perseus region and that at least a good part of the A stars share in the motion of the helium stars.[1] It thus seems not unlikely that further investigation by these methods will give *good determinations of the parallaxes of all the B stars and of a large number of the A stars.*

For the later types the beautiful method proposed by Adams and Kohlschütter[2] and developed in detail by Adams[3] has recently opened the prospect of extensive determination of parallax by spectroscopic means. All this awakens the hope that we are at last on the way toward a wholesale determination of the third co-ordinate—distance—the lack of which has been the main obstacle in the way of substantial knowledge of the structure of the stellar system.

The extreme importance of the matter makes it desirable to obtain for the B stars the most reliable values of the parallax possible. I therefore give another solution for r_u in which the stars less satisfactorily observed are not altogether neglected.

For the Scorpius-Centaurus region all the stars were used, whatever their parallaxes; for the Orion region the stars with parallaxes $<0\rlap{.}''0070$ were neglected, because it was feared that the remaining errors in the very small parallaxes would influence too greatly the values for t obtained by (43). For the first region the stars were subdivided into six groups,[4] the limits of the parallax being respectively: $100\pi \leqq 0\rlap{.}''49$; $0\rlap{.}''50$–$0\rlap{.}''69$; $0\rlap{.}''70$–$0\rlap{.}''86$; $0\rlap{.}''87$–$1\rlap{.}''04$; $1\rlap{.}''05$–$1\rlap{.}''27$; $1\rlap{.}''28$–$2\rlap{.}''24$. For the Orion region the limits were the same, but

[1] *Trans. Internat. Solar Union*, **3**, 220, 1911.

[2] *Mt. Wilson Contr.*, No. 89; *Astrophysical Journal*, **40**, 385, 1914.

[3] *Mt. Wilson Comm.*, Nos. 23–25; *Proc. Nat. Acad. Sci.*, **2**, 143, 147, 152, 1916.

[4] In Table XV the first two had to be combined.

the first two groups are wanting. The first four columns of Tables XIV and XV contain, respectively, the averages of $100\ \pi$, $100\ \tau$, $100\ r$, and of $\bar{t}$ computed from $\bar{\tau}$ and $\bar{\pi}$ by (43). The fifth column contains the probable amount of $100\,\tau$ freed from observational

TABLE XIV

AVERAGE VALUES OF t, ETC.

Orion Region							Scorpius-Centaurus Region						
$100\,\bar{\pi}$	$100\,\bar{\tau}$	$100\,\bar{r}$	$\bar{t}$	$100\,r_\tau$	r_t	No.	$100\,\bar{\pi}$	$100\,\bar{\tau}$	$100\,\bar{r}$	$\bar{t}$	$100\,r_\tau$	r_t	No.
							(0″.32)	(0″.61)	(0″.62)	(7.66)	(0″.000)	(0.0)	(8)
							0.62	0.44	0.55	3.32	0.000	0.0	14
0″.77	0″.62	0″.48	3.70	0″.210	1.3	21	0.78	0.57	0.51	3.50	0.000	0.0	19
0.94	1.04	0.56	5.17	0.677	3.4	23	0.96	0.61	0.48	3.02	0.187	0.9	16
1.13	0.55	0.58	2.32	0.000	0.0	21	1.15	0.76	0.50	3.11	0.403	1.65	20
1.52	0.83	0.44	2.88	0.546	1.7	12	1.59	0.83	0.59	2.55	0.379	1.1	22
1.04	0.761	0.526	3.63	0.370	1.68 1.64	77	1.07	0.662	0.529	3.07	0.181	0.80 0.79	91

TABLE XV

SAME AS TABLE XIV BUT FOR $100\ r \leqq 0''.40$

Orion Region							Scorpius-Centaurus Region						
$100\,\bar{\pi}$	$100\,\bar{\tau}$	$100\,\bar{r}$	$\bar{t}$	$100\,r_\tau$	r_t	No.	$100\,\bar{\pi}$	$100\,\bar{\tau}$	$100\,\bar{r}$	$\bar{t}$	$100\,r_\tau$	r_t	No.
							0″.45	0″.33	0″.35	2.40	0″.000	0.0	2
0″.77	0″.58	0″.35	3.62	0″.343	2.1	10	0.79	0.28	0.31	1.72	0.000	0.0	5
0.91	0.50	0.26	2.40	0.332	1.7	5	0.96	0.30	0.32	1.45	0.000	0.0	8
1.12	0.32	0.33	1.35	neg.	0.0	6	1.13	0.53	0.32	2.22	0.314	1.3	6
1.57	0.87	0.31	2.90	0.666	2.0	7	1.89	1.45	0.37	4.02	1.168	2.9	4
1.07	0.582	0.321	2.74	0.373	1.64 1.55	28	1.07	0.538	0.329	2.18	0.314	1.38 0.78	25

error, as follows: Admitting that the values of τ are distributed substantially according to an error-curve (see Table XII) the probable amount $100\ r_\tau$ corresponding to $100\ \tau$ was found by multiplying the average amount by the factor 0.845. Freed from observational error we therefore evidently have

$$100\ r_\tau = 100\sqrt{(0.845\ \bar{\tau})^2 - \bar{r}^2}. \tag{45}$$

In a few cases, the quantity under the radical being negative, r_τ was assumed to be zero. With the aid of r_τ we find the probable amount, $r_t=r_u$ of t, freed from observational error by

$$r_t=\frac{r_\tau}{0.212\,\pi}, \tag{46}$$

which evidently follows from (43). These quantities are given in the sixth column. In the seventh are the numbers of stars included in each average. The last line of each table shows the total averages; for r_t two values are given, the first, from the total averages of r_τ and π by (46), the second, from the separately computed values of r_t with weights proportional to the number of stars. Except for the last value in Table XV, the agreement is almost exact; I adopt the means.

Supplementing these results with those obtained by the same method when we limit ourselves still more closely to the best-observed stars, we have the following values for $r_t=r_u$:

Orion				Scorpius-Centaurus			
100 r	Reference	r_t	No.	100 r	Reference	r_t	No.
		km				km	
All values..	Table XIV	1.66	77	$\leqq 0\overset{''}{.}80$..	Table XIV	0.80	91
$\leqq 0\overset{''}{.}40$....	(44)	1.80	28	$\leqq 0.40$..	(44)	1.12	25
$\leqq 0\overset{''}{.}40$....	Table XV	1.60	28	$\leqq 0\overset{''}{.}40$..	Table XV	1.08	25
$\leqq 0.30$....	....	1.56	9	$\leqq 0.30$..	....	0.75	7

from which finally the *adopted values*

$$\left.\begin{array}{ll}\text{Orion region} & \boldsymbol{r_u=\pm 1.67\ \text{km}}\\ \text{Scorpius-Centaurus region} & \boldsymbol{r_u=\pm 1.00\ \text{km}}\end{array}\right\} \tag{47}$$

12. REMARKS

Remark 1.—It has been implicitly assumed in what precedes that the errors in the parallaxes do not sensibly influence the results for r_u. I have convinced myself that such is really the case.

Remark 2.—The accuracy of the determination of r_u certainly is still of a rather low order. I have refrained from computing

probable errors, because the data are hardly adequate for a good determination. It is easy to obtain an upper limit for r_u by neglecting the observational errors altogether. From Table XV we find, since $r_t = 0.845\,\bar{t}$,

$$\text{Upper limit for } r_u \begin{cases} \text{Orion region} & = \pm 2.3 \text{ km} \\ \text{Scorpius-Centaurus region} & = \pm 1.8 \text{ km} \end{cases} \qquad (48)$$

With these numbers before us it cannot be doubted that the real values of r_u do not materially exceed the adopted values (47); the chances are that the latter are somewhat too high.

In consideration of the remaining uncertainty, however, I have carried through the computations that follow on the two extreme suppositions

$$r_u = \pm 2.5 \text{ km and } r_u = 0.0 \text{ km.} \qquad (49)$$

This will make a subsequent correction easy, if we find means of improving our values by the addition of well-observed fainter stars, by the inclusion of later observations, or by the discovery that the A stars for the region under consideration show the same motions, etc. For the present we shall definitely adopt the parallaxes corresponding to the first of (47); these will be obtained by interpolating between the two solutions based on (49).

Remark 3.—The value now derived for the Scorpius-Centaurus region is considerably smaller than $r_u = \pm 2.1$ found in *Mount Wilson Contribution* No. 82, where fortunately it does not play nearly as conspicuous a part as it will here. This was obtained in a less direct way. As far as I can see, there are two causes for the divergence, both of which tend to give too large a value: First, in *Mount Wilson Contribution* No. 82, all stars for which the probable error of the position angle of the proper motion, due to observational error, exceeds 10°, have been omitted. It can be shown that a small systematic error was thus introduced, the effect of which is an increase in the value for r_u. Secondly, in that same paper there was no limitation to stars below galactic longitude 320°. The consequence must be the introduction of a relatively larger number of stars not belonging to the group. Avoiding the first source of error by not limiting the values of r_p and excluding only the four

extreme values,[1] I find by the method of *Mount Wilson Contribution* No. 82, $r_u = \pm 1.3$ km (135 stars), which is in fairly good agreement with (47). By eliminating the second source of error as well, we should no doubt find the agreement quite satisfactory.

Remark 4.—The value (42) obtained from the best radial velocities gives for the probable amount of u, including errors of observation, ± 3.6 km. Since we adopt $r_u = 1.67$ km, the probable observational error of the radial velocities is ± 3.2 km. For the Scorpius-Centaurus region the corresponding quantity is ± 2.8 km.

[1] If we do not exclude these, the value of r becomes still smaller.

[*To be continued*]

ON THE ORBIT OF THE SPECTROSCOPIC BINARY 42 CAPRICORNI

By JOSEPH LUNT

Two plates of the spectrum of this star (H.R. 8283, $\alpha = 21^h 36^m.1$, $\delta = -14° 29'$ [1900]; Mag., 5.28; Type K) taken on October 2 and October 5, 1917, gave radial velocities of -17.7 and $+7.2$ km per second, respectively. As its variable velocity appeared not to have been announced previously, fifteen further plates were secured during October and November, and these on measurement gave the results shown in Table I (p. 105).

The measures were made on the Hartmann spectro-comparator using plate 4903 of α Tauri as standard, the shift on this plate being taken as $+82.34$ km.

The observations were plotted on millimeter paper to the scale 30 mm = 1 day, 10 mm = 2 km, and the following provisional elements of the orbit were derived by graphical methods:

$$P = 13.25 \text{ days}$$
$$T = \text{J.D. } 2421525.16$$
$$\omega = 175°$$
$$e = 0.20$$
$$K = 22.75 \text{ km per second}$$
$$a \sin i = 4{,}061{,}000 \text{ km}$$
$$V_0 = -3.0 \text{ km per second}$$

Correction for solar motion $= +7.3$ km

The system is therefore receding from us with a radial velocity of 4.3 km, on the assumption that the solar motion is toward $18^h, +30°$ at 20 km per second.

Fig. 1 shows the observations compared with the theoretical curve. The figures beside the circles denoting the observations show the number of periods to be applied to the dates given above the curve to obtain the date of observation.

Table II shows the observations arranged in order of phase and the differences between the observed velocities and those derived

from a curve of theoretical values corresponding to the foregoing elements.

TABLE I

Plate	Date 1917	Sidereal Time	Correction to Sun	Radial Velocity
5080.....	Oct. 2	22^h23^m	−22.22 km	−17.7 km
5084.....	5	22 15	23.23	+ 7.2
5090.....	17	22 31	26.81	− 0.7
5092.....	23	22 14	27.98	+ 4.0
5094.....	24	22 11	28.16	− 3.5
5096.....	25	22 31	28.38	−11.1
5098.....	26	22 27	28.54	−22.4
5100.....	27	22 27	28.69	−32.2
5101.....	31	22 44	29.29	+ 5.0
5103.....	Nov. 3	22 47	29.63	+16.4
5105.....	6	23 27	29.97	− 0.6
5106.....	7	23 35	30.04	− 7.4
5107.....	8	23 16	30.09	−22.3
5108.....	9	23 7	30.12	−29.5
5110.....	10	23 30	30.20	−23.7
5112.....	12	23 37	30.27	− 2.9
5115.....	16	23 49	30.28	+13.9

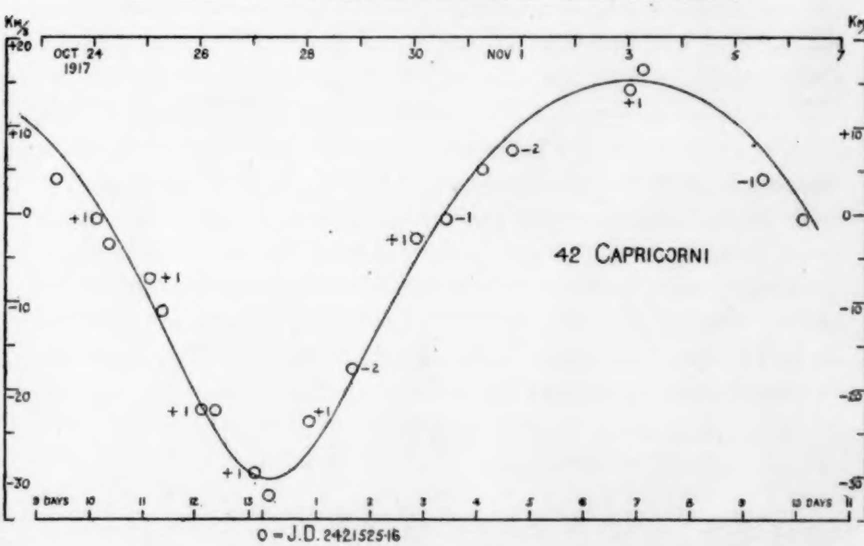

FIG. 1.—Velocity-curve of 42 Capricorni

Further observations next May are contemplated in order to determine the period more accurately, as the present observations extend over only three periods.

Campbell[1] remarks that "it is a striking fact that there are no known binaries of the G, K, and M types (excepting possibly H.R. 142, 13 Ceti) whose periods are less than twenty days." The short period (13.25 days) of this binary is therefore of interest.

TABLE II

Date 1917	G.M.T.	Phase	Observed Velocity	Computed Velocity	O−C
Oct. 27	6^h51^m	$0^d.1187$	−32.2 km	−30.3 km	−1.9 km
Nov. 10	6 59	0.8743	−23.7	−27.2	+3.5
Oct. 2	8 26	1.6847	−17.7	−18.8	+1.1
Nov. 12	6 58	2.8736	− 2.9	− 5.2	+2.3
Oct. 17	8 34	3.4403	− 0.7	+ 0.5	−1.2
Oct. 31	6 52	4.1194	+ 5.0	+ 5.8	−0.8
Oct. 5	8 6	4.6678	+ 7.2	+ 9.3	−2.1
Nov. 16	6 54	6.8708	+14.1	+15.2	−1.1
Nov. 3	6 43	7.1132	+16.4	+15.1	+1.3
Oct. 23	6 54	9.3708	+ 4.0	+ 7.3	−3.3
Nov. 6	7 12	10.1750	− 0.6	+ 1.2	−1.8
Oct. 24	6 47	10.3663	− 3.5	− 1.2	−2.3
Nov. 7	7 16	11.1361	− 7.4	− 9.7	+2.3
Oct. 25	7 3	11.3771	−11.1	−12.8	+1.7
Nov. 8	6 53	12.1201	−22.3	−21.9	−0.4
Oct. 26	6 55	12.3715	−22.4	−24.8	+2.4
Nov. 9	6 40	13.1111	−29.5	−29.9	+0.4

The type could not be determined precisely from the limited region of spectrum photographed, but it appears to agree closely with the solar spectrum in the region examined. Harvard classifies it as of Type K.

Messrs. Woodgate and Baines assisted by exposing ten of the plates, which were taken with the four-prism spectrograph of the 24-inch Victoria telescope, using the short camera. The measures were made by the writer.

ROYAL OBSERVATORY, CAPE OF GOOD HOPE
November 26, 1917

[1] "Second Catalogue of Spectroscopic Binary Stars," *Lick Observatory Bulletin*, No. 181, **6**, 35, 1910.

Minor Contributions and Notes

ON CHANGES OF THE WAVE-LENGTHS OF CERTAIN LINES IN STELLAR SPECTRA DEPENDING UPON THE TYPE

The object of this research was to examine to what extent stellar lines measured in spectrograms taken with the McClean telescope show relative changes, with type, of the kind already announced by Albrecht.[1]

The spectrograms were measured with Halm's wave-length machine.[2] Of the four stars, α Canis Majoris, α Canis Minoris, α_2 Centauri, and α Boötis, representing the spectral classes A, F, G, and K, respectively 10, 5, 8, and 10 plates have been measured. All the spectrograms contain a comparison spectrum of iron. One setting only was made on both the iron lines and the stellar lines. As far as possible the same iron and stellar lines were measured for each type.

The mean of the measured Fe lines for each of the four stars was formed separately and the four means were then combined into one, weighted according to the number of plates measured for every star. The differences of the observed wave-lengths were then platted in the form of a graph, and from the smoothed curve the corrections were obtained which are to be added to the observed mean Fe lines in order to reduce them to the Rowland system. The mean Fe lines for each star were then reduced to this system. The probable error of a measurement of one Fe line on a single plate was found to be ± 0.015 A.

With regard to the stellar lines, for each of the four stars separately the measured lines were first combined into means, and then the correction of every stellar line to the Rowland system was taken from the curve mentioned above. Finally the wave-lengths of the

[1] *Astrophysical Journal*, **33**, 130, 1911.

[2] *Annals of the Cape Observatory*, **10**, Part I.

λ	A	F	G	K
4191...	......	.77	.72	.79
90...	......	.50	.55	.61
99...	......	.38	.35	.37
4200...	......	......	......	.21
02...	.28	.32	.26	.27
04...	......	.20	.23	.19
06...	......	......	.85	.88
10...	......	.57	.56	.60
15...	.77	.73	.70	.83
16...	......	......	......	.36
19...	......	.56	.53	.59
22...	......	.43	.41	.42
25...	......	......	......	.60
26...	......	.89	.94	27.10
27...	.83	.64	.61	......
35...	......	.16	.38	.40
36...	.15	......	.19	.12
38...	......	.94	39.02	39.04
40...	......	.00	.03	.06
42...	.57	......	.66	......
43...	......	......	.56	.67
45...	......	.46	.48	.48
47...	......	.01	.01	.03
		47.00	46.97	46.96
47...	......	......	.62	.64
48...	......	......	......	.52
50...	......	.32	.33	.36
50...	......	.99	51.00	51.02
52...	......	......	......	.51
54...	......	.54	.56	.55
		.52	.50	.49
58...	......	......	......	.56
60...	......	......	......	.31
60...	.67	.73	.57	.81
	......	.64	.67	.72
68...	......	......	......	.13
		67.86	67.90	67.96
71...	.37	.34	.44	.55
71...	.96	.99	.95	72.07
74...	......	.98	.96	75.01
		.97	.95	.94
88...	......	.17	.20	.26
		.10	.15	.16
89...	......	......	......	.95
91...	......	......	.20	.28
94...	.33	.33	.32	.34
4299...	......	.40	.27	......
4308...	.13	.10	.10	.17
14...	......	......	......	.43
		.34	.36	.36
15...	.27	.34	.24	.21
	......	.18	.16	.15
18...	......	.85	.92	.93
21...	......	.03	.02	20.99
		.01	20.96	20.91
25...	......	.17	.27	.30
		.19	.19	.23
26...	.00	25.99	.02	.06
4333...	......	......	......	.02

λ	A	F	G	K
4335...	......	......	......	.09
		34.99	35.06	35.10
37...	......	......	.29	.29
40...	.71	.78	.73	.68
	......	.66	.66	.65
44...	......	......	......	.64
		.49	.59	.62
52...	.04	.06	.03	......
	......	.02	.01	.01
52...	......	......	.96	53.02
		.94	.96	.98
58...	......	.76	.84	.86
59...	......	.84	.80	.83
67...	......	.91	.84	......
69...	......	......	.89	.96
71...	......	......	.36	.41
76...	......	......	.12	.18
79...	......	......	......	.41
83...	.76	.76	.78	.82
84...	......	......	......	.94
95...	.24	.25	.30	.32
	......	.24	.26	.27
4399...	.98	.90	.84	......
	......	.95	.92	.89
4401...	......	......	.70	.69
04...	.97	.97	05.00	.97
07...	......	......	.84	.90
08...	......	......	.65	.61
15...	.27	.31	.33	.40
25...	......	.62	.66	......
		.59	.64	.88
27...	......	.47	.46	.48
35...	......	.17	.22	.25
		.16	.19	.24
42...	......	......	......	.59
47...	......	......	.94	.95
55...	......	.03	.99	.99
59...	......	.32	.29	.33
64...	......	.71	.77	.82
		.73	......	.90
66...	......	.75	.76	.80
68...	.71	.68	.68	.73
	......	.65	.67	.70
69...	......	.53	.56	.63
		.49	.56	.62
72...	......	.06	71.98	.03
76...	......	.26	.24	.29
81...	.45	.43	.38	......
82...	......	.38	.37	.41
4494...	......	.76	.72	.75
4501...	.51	.47	.39	.46
15...	.63	.50	.49	......
22...	.84	.84	.82	.89
28...	......	.81	.81	......
33...	......	.22	.29	.29
34...	.25	......	.33	......
36...	......	......	.02	.06
49...	.67	.83	.77	.84

stellar lines thus obtained were corrected for the radial velocity and for the earth's motion.

The table on page 138 shows the final normal wave-lengths of the stellar lines for each of the four types and in addition the wave-lengths derived by Albrecht for those lines of similar types which are common to the two investigations. Where the values are bracketed the lower ones are Albrecht's.

The agreement with Albrecht's wave-lengths is in general fairly good, the more pronounced progressions in Albrecht's values, from one type to another, being generally confirmed by the present observations.

J. VOÛTE

ROYAL OBSERVATORY, CAPE OF GOOD HOPE
December 12, 1917

THE OBSERVATORY OF POULKOVA

Disquieting rumors regarding the safety of the observatory at Poulkova reached this country in brief press dispatches during the early winter. We are now happy to be able to state[1] that the dangers due to the civil strife had thus far been passed without serious damage to that famous institution. It is, however (as it should be!), very unusual for an astronomical observatory to be under artillery fire, and it can hardly be an indiscretion to give our readers the following particulars:

Rumors had reached the astronomical colony at Poulkova, which lies about 20 km southwest of Petrograd, that bodies of Cossacks were coming toward the capital to restore the ministry of M. Kerensky. The garrison from the village of Zarskoje Selo (site of the summer residence of the former emperor) presently arrived at Poulkova, having been driven out by the Cossacks. On November 11 soldiers from the garrison of Petrograd, with the Red Guard and with artillery, arrived for the relief of that body of troops.

With a zeal which we may imagine must have seemed to the astronomers in excess of its wisdom, those forces from the capital

[1] As of November 23, 1917.

surrounded the observatory on all sides and established batteries of artillery within 400 or 500 meters on both sides of the main building.

Between one o'clock and five-thirty on the afternoon of November 12 the observatory was under an intense artillery fire from the Cossacks. Fortunately, none of the instruments were damaged, although a shell burst beside the brick foundation of the dome of the large astrographic telescope. Many holes were made in the dome of the great refractor and in the roof of the director's office. The wall of the seismological station was pierced, and we may believe that the instruments registered their largest earthquake. Of course there was much damage to windows. The ground was torn up by numerous holes of a diameter of a meter. The Cossacks retired on the following day.

The director of the observatory had been able in the preceding days to foresee the dangers of the situation, and had removed the 30-inch objective, as well as the objectives of the other instruments. By a singular good fortune no one of the personnel of the observatory or colony was injured during these exciting events.

It was felt that there was no guaranty that such events might not recur under the present conditions in Russia. Men of science everywhere will certainly wish to join with us in our satisfaction that no more serious damage was done and in offering our congratulations on this escape to Director Belopolsky and his staff, with the hope that the admirable work carried on at the observatory during the last eighty years may continue unimpaired, despite the political and economic changes through which Russia is passing.

F.

March 1918